

model III:

Length of the queue $L_q = P_s \cdot \frac{\rho}{(1-\rho)^2}$

$$2) \text{ where } P_s = \frac{\left[\frac{\lambda}{\mu}\right]^S}{S!} P_0$$

$$\therefore L_s = \frac{\lambda}{\mu} + L_q$$

$$W_q = \frac{L_q}{\lambda}$$

$$W_s = \frac{L_s}{\lambda}$$

$\therefore$ The mean No. of Individuals who actually wait is given by

$$L_{LTD} = \frac{1}{1-\rho}$$

$\rightarrow$ The mean waiting time in a queue for car who actually wait is given by

$$W = \frac{1}{S\mu - \lambda}$$

$$\rightarrow \text{probability } (W > 0) = \frac{P_s}{1-\rho}$$

$\rightarrow$ probability that there are some one waiting

$$\frac{P_s \cdot \rho}{1-\rho}$$

$$\rightarrow \text{Avg no. of idle servers} = S - [\text{avg no. of customers served}]$$

$$\rightarrow \text{efficiency of [M/M/S] model} = \frac{\text{Avg no. of customers served}}{\text{Total no. of customers served}}$$

Project Management

A project consist of inter related activities which are to be executed in a cert certain orders before the entire task is completed.

→ The activities are interrelated in a logical sequence which is known as ~~preceeded~~ ^{predecessor} sequence.

* The project is represented in the form of Nlws for the purpose of analytical treatment to get solutions for scheduling & controlling it's activities.

Techniques:

1) cpm: (critical path method) (longest path.

2) PERT: (project evaluation review technique)

Phases of project managment:

→ planning

→ scheduling

→ controlling

cpm (critical path method):

1) consider the details of a project as shown in the table?

→ consider the details of a project as shown in the table

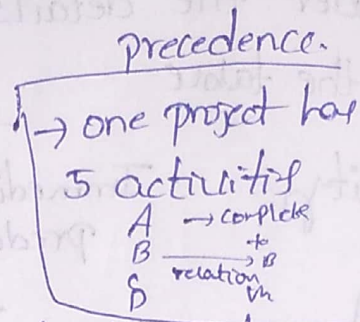
Activity	Immediate predecessor(s)	duration (months)
A	-	2
B	-	5
C	-	4
D	B	5
E	A	7
F	A	3
G	B	3
H	C, D	6
I	C, D	2
J	E	5
K	F, G, H	4
L	F, G, H	3
M	I	12
N	J, K	8

a) construct cpm N/W

b) Determine the critical path & project completion time.

c) compute total floats & free floats for non-critical activities.

- Banz concept
- Project management
 - Phases of Project Management
 - Guidelines for N/w construction
 - Critical path



* A project consists of interrelated activities which are to be executed in a certain order before the entire task is completed.

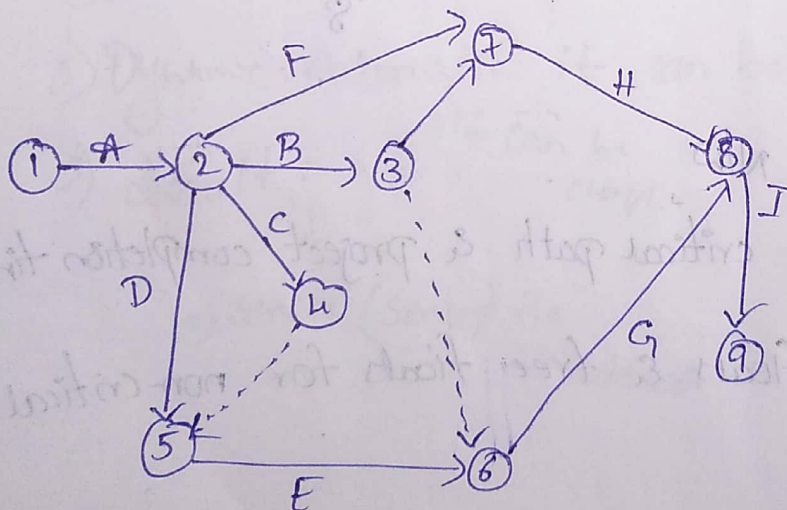
* The activities are interrelated in a logical sequence which is known as precedence Relationship.

* Project is represented in the form of a N/w for the purpose of analytical treatment to get solutions for scheduling & controlling its activities.

Techniques

cpm → critical Path method

PERT → project evaluation & Review technique.



→ Guidelines for N/w construction

- 1) ① ② --- ⑨ nodes
- 2) A, B --- i activities

3) starting event & ending event of an activity are called tail " " head.
 ① → ②
 tail head

4) each & every activity has a unique tail event (starting)

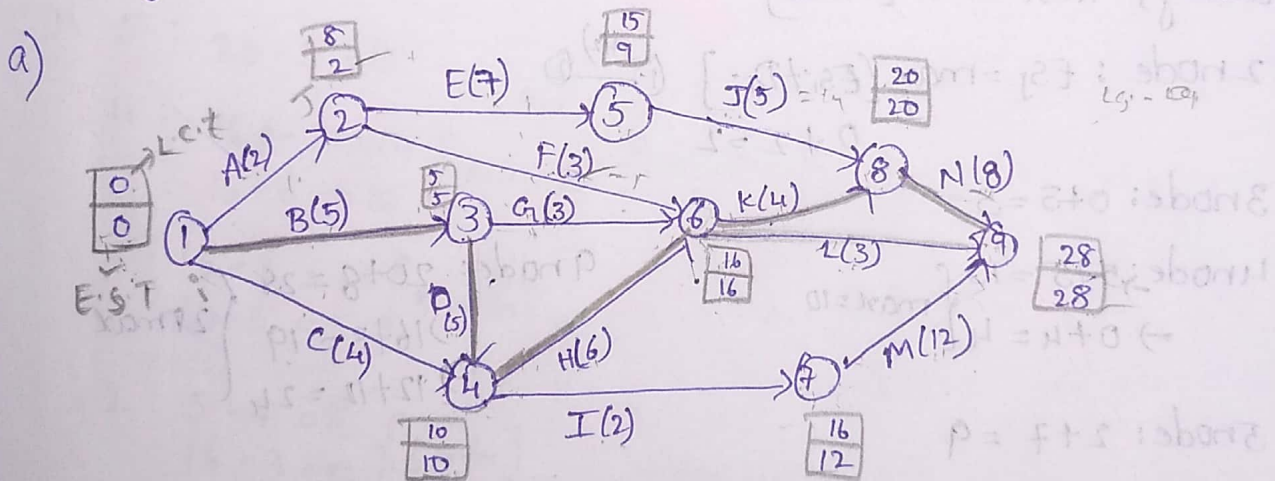
5) " " " " " " head " (ending)

6) Each activity has two nodes, there will be

7) NO activity should be represented by more than one activity in the Nlw.

8) NO two activities should have the same starting node and the same ending node.

9) Dummy activity is an imaginary activity indicating precedence relationship only. Duration of a dummy activity is zero.



→ critical path

1) The project nlw is the longest path in the Nlw.

2) we have follow — This can be satisfied by simply out all the possible paths from the start node of the project to the end node of the project & then selecting the path with the maximum sum of activity forms on that path.

Two phases

→ Determine earliest start time [ES] of ^{all} the nodes.
[called as forward pass]

→ Determine latest completion ~~times~~ times [LC] of various nodes [called as backward pass]

find E.S.T & L.C.T of each & every node:-

1) EST [Earliest start time]

formula: $ES_j = \max_i (ES_i + D_{ij})$

j = ending activity

i = starting activity

ES_j = Earliest start time of ending activity

D = duration

EST:

1) always first zero. [in node]

2) 2 node: $ES_j = \max_i (ES_i + D_{ij})$

$0 + 2 = 2$

3 node: $0 + 5 = 5$

4 node: $5 + 5 = 10$
→ $0 + 4 = 4$ { max = 10

5 node: $2 + 7 = 9$

6 node: $2 + 3 = 5$
→ $5 + 3 = 8$
→ $10 + 6 = 16$ { 16 max

7 node: $10 + 2 = 12$

8 node: $9 + 5 = 14$
→ $16 + 4 = 20$ { 20 max

9 node: $20 + 8 = 28$
→ $16 + 3 = 19$
→ $12 + 12 = 24$ { 28 max

Latest completion time: [backward]
starting this process starts with node 9

Determination of LCT (formula)

$$LC_i = \min (LC_j - D_{ij})$$

latest completion starting node = LC_i

" " ending node = LC_j

LCT
node 9 = 28

node 8 : $LC_j - D_{ij}$

$$LC_9 - D_{89}$$

$$\Rightarrow 28 - 8 = 20$$

node 7 :

$$LC_j - D_{ij}$$

$$28 - 12 = 16$$

node 6 : $28 - 3 = 25$
 $20 - 4 = 16$ { min = 16 }

" 5 : $20 - 5 = 15$

4 : $16 - 2 = 14$
 $16 - 6 = 10$ { min = 10 }

3 : $16 - 3 = 13$
 $10 - 5 = 5$ { 5 min }

2 : $15 - 7 = 8$
 $16 - 3 = 13$ { 8 min }

1 : $8 - 2 = 6$
 $5 - 5 = 0$
 $10 - 4 = 6$ { 0 min }

Condition for critical path:-

$$ES_i = LC_i$$

$$ES_j = LC_j$$

$$ES_j - ES_i = LC_j - LC_i = D_{ij}$$

$$\Rightarrow 1-3-4-6-8-9$$

critical path B-D-H-K-N

duration:

$$5+5+6+4+8 = 28$$

c) Now we calculating total floats & free floats of NON-critical path.

Total floats: It is the amt of time that the completion time of an activity can be ^{delayed} displayed without affecting the project completion time.

formula: $TF_{ij} = LC_j - ES_i - D_{ij}$

Activity 1-2
(A)

$$\Rightarrow 8 - 0 - 2 = 6$$

$$(B) \Rightarrow 5 - 0 - 5 = 0$$

$$(C) \Rightarrow 10 - 0 - 4 = 6$$

$$(D) \Rightarrow 10 - 5 - 5 = 0$$

$$(E) \Rightarrow 15 - 2 - 7 = 6$$

$$(F) \Rightarrow 16 - 2 - 3 = 11$$

$$(G) \Rightarrow 16 - 2 - 3 = 11$$

$$(H) \Rightarrow 16 - 10 - 6 = 0$$

$$(I) \Rightarrow 16 - 10 - 2 = 4$$

$$(J) \Rightarrow 20 - 9 - 5 = 6$$

$$(K) \Rightarrow 20 - 16 - 4 = 0$$

$$(L) \Rightarrow 28 - 16 - 3 = 9$$

$$(M) \Rightarrow 28 - 12 - 12 = 4$$

$$(N) \Rightarrow 28 - 20 - 8 = 0$$

Summary of total floats & free floats

Activity (i,j)	Duration (D _{ij})	Total float (TF _{ij})	Free float (FF _{ij})
A 1-2	2	6	0
B 1-3	5	0	0
C 1-4	4	6	6
D 3-4	5	0	0
E 2-5	7	6	0
F 2-6	3	11	11
G 3-6	3	8	8
H 4-6	6	0	0
I 4-7	2	4	6
J 5-8	5	6	0
K 6-8	4	0	9
L 6-9	3	9	4
M 7-9	12	4	0
N 8-9	8	0	0

Free float

$$FF_{ij} = ES_j - ES_i - D_{ij}$$

activity A 1-2

$$\Rightarrow 2 - 0 - 2 = 0$$

Here T.F & FF values are zero. This property we can determine critical activities

$$(B) \Rightarrow 5 - 0 - 5 = 0$$

$$(C) \Rightarrow 10 - 0 - 4 = 6$$

$$(D) \Rightarrow 10 - 5 - 5 = 0$$

$$(E) \Rightarrow 9 - 2 - 7 = 0$$

$$(F) \Rightarrow 16 - 2 - 3 = 11$$

$$(G) \Rightarrow 16 - 5 - 3 = 8$$

$$(H) \Rightarrow 16 - 10 - 6 = 0$$

$$(I) \Rightarrow 12 - 10 - 2 = 0$$

$$(J) \Rightarrow 20 - 9 - 5 = 6$$

$$(K) \Rightarrow 20 - 16 - 4 = 0$$

$$(L) \Rightarrow 28 - 16 - 3 = 9$$

$$(M) \Rightarrow 28 - 12 - 12 = 4$$

$$(N) \Rightarrow 28 - 20 - 8 = 0$$

2) consider the details of a project as shown in below.

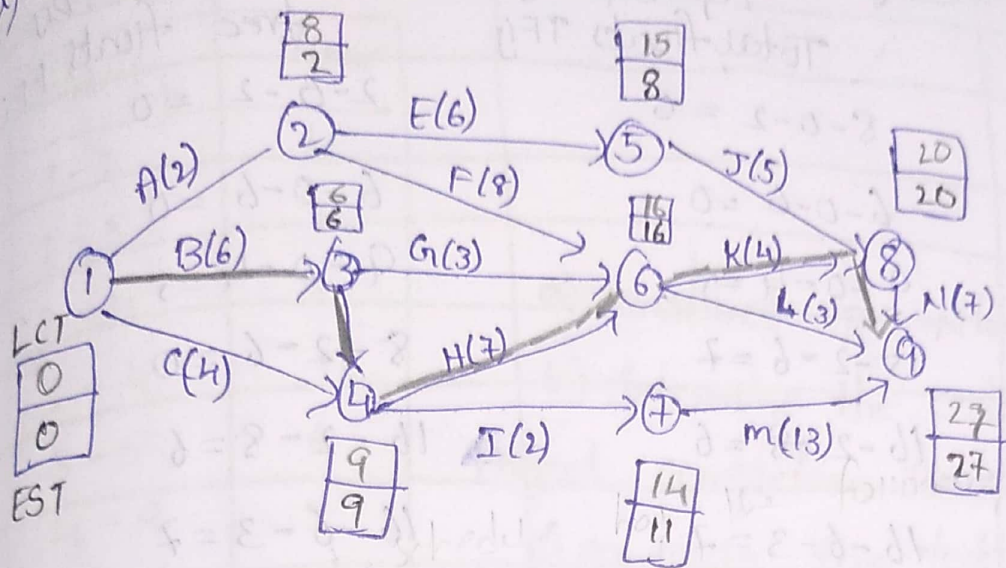
Activity	Immediate Predecessor(s)	duration
A	-	2
B	-	6
C	-	4
D	B	3
E	A	6
F	A	8
G	B	3
H	C D	7
I	C D	2
J	E	5
K	F G H	4
L	F G H	3
M	I	13
N	J K	7

a) construct cpm n/w.

b) determine the critical path & project completion time

c) compute total floats & free floats for non-critical activities.

a) construct CPM N/W



b) Earliest com starting time : [forward] [$\rightarrow$] [1 to 9]

formula: $ES_j = \max_i (ES_i + D_{ij})$

Node 1: 0

2 : $0 + 2 = 2$

3 : $0 + 6 = 6$

4 : $0 + 4 = 4$
 $6 + 3 = 9$ } max=9

5 : $2 + 6 = 8$

6 : $2 + 8 = 10$
 $6 + 3 = 9$
 $9 + 7 = 16$ } max=16

7 : $9 + 2 = 11$

8 : $16 + 4 = 20$
 $8 + 5 = 13$ } max=20

9 : $11 + 13 = 24$
 $20 + 7 = 27$
 $16 + 3 = 19$ } max=27

Latest completion time:

[Backward] [$\leftarrow$] [9 to 1]

formula: $LC_j = \min_i (LC_i - D_{ij})$

Node 9: 27

8 : $27 - 7 = 20$

7 : $27 - 13 = 14$

6 : $20 - 4 = 16$
 $27 - 3 = 24$ } min=16

5 : $20 - 5 = 15$

4 : $14 - 2 = 12$
 $16 - 7 = 9$ } min=9

3 : $16 - 3 = 13$
 $9 - 3 = 6$ } min=6

2 : $15 - 6 = 9$
 $16 - 8 = 8$ } min=8

1 : $8 - 2 = 6$
 $6 - 6 = 0$
 $9 - 4 = 5$ } min=0

Summarizing the total floats & free floats

Activity	$LC_j - ES_i - D_{ij}$ Total floats TF_{ij}	$ES_j - ES_i - D_{ij}$ free floats FF_{ij}
1-2	$8 - 0 - 2 = 6$	$2 - 0 - 2 = 0$
1-3	$6 - 0 - 6 = 0$	$6 - 0 - 6 = 0$
1-4	$9 - 0 - 4 = 5$	$9 - 0 - 4 = 5$
2-5	$15 - 2 - 6 = 7$	$8 - 2 - 6 = 0$
2-6	$16 - 2 - 8 = 6$	$16 - 2 - 8 = 6$
3-6	$16 - 6 - 3 = 7$	$16 - 6 - 3 = 7$
3-4	$9 - 6 - 3 = 0$	$9 - 6 - 3 = 0$
4-6	$16 - 9 - 7 = 0$	$16 - 9 - 7 = 0$
4-7	$14 - 9 - 2 = 3$	$11 - 9 - 2 = 0$
5-8	$20 - 8 - 5 = 7$	$20 - 8 - 5 = 7$
6-8	$20 - 16 - 4 = 0$	$20 - 16 - 4 = 0$
6-9	$27 - 16 - 3 = 8$	$27 - 16 - 3 = 8$
7-9	$27 - 11 - 13 = 3$	$27 - 11 - 13 = 3$
8-9	$27 - 20 - 7 = 0$	$27 - 20 - 7 = 0$

∴ The critical path: [longest path]

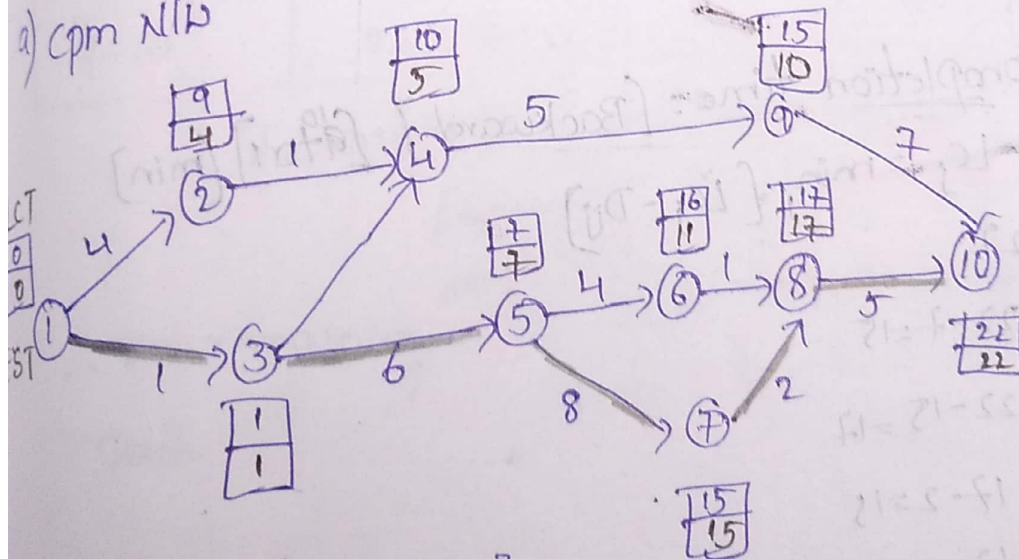
1-3-4-6-8-9

consider the details of a project as shown in the below.

Activity	time
1-2	4
1-3	1
2-4	1
3-4	1
3-5	6
4-9	5
5-6	4
5-7	8
6-8	1
7-8	2
8-10	5
9-10	7

- construct cpm N/w.
- determine the critical path & project completion time.
- compute the total float & free float for non-critical activities.

a) cpm N/w



critical path : [longest]

1-3-5-7-8-10

EST [Earliest starting time [forward] [→] [1 to 10]]

formula: $ES_j = \max_i (ES_i + D_{ij})$

node 1: 0

2: $0 + 4 = 4$

3: $0 + 1 = 1$

4: $4 + 1 = 5$ { 5 max

1: $1 + 1 = 2$

5: $1 + 6 = 7$

6: $7 + 4 = 11$

7: $7 + 1 = 8$ { 10 max

8: $7 + 8 = 15$

9: $15 + 2 = 17$

10: $8 + 1 = 9$ { max 17

11: $5 + 5 = 10$

12: $10 + 7 = 17$

13: $17 + 5 = 22$ { 22 max

Latest completion time: [Backward] [10 to 1] [min]

formula: $LC_j = \min_i [LC_i - D_{ij}]$

Node 10: 22

9: $22 - 7 = 15$

8: $22 - 15 = 7$

7: $17 - 2 = 15$

6: $17 - 1 = 16$

5: $16 - 4 = 12$

15: $15 - 8 = 7$ { min 7

4: $15 - 5 = 10$

3: $10 - 1 = 9$ { min 1

7: $7 - 6 = 1$

node 2: $10 - 1 = 9$

node 1: $9 - 4 = 5$

1: $1 - 1 = 0$ { 0 min

Summarizing the total floats & free floats

Activity	$LS_j - ES_i - D_{ij}$ Total float TF _{ij}	$ES_j - ES_i - D_{ij}$ free float FF _{ij}
1-2	$9 - 0 - 4 = 5$	$4 - 0 - 4 = 0$
1-3	$1 - 0 - 1 = 0$	$1 - 0 - 1 = 0$
2-4	$10 - 4 - 1 = 5$	$5 - 4 - 1 = 0$
3-4	$10 - 1 - 1 = 8$	$5 - 1 - 1 = 3$
3-5	$7 - 1 - 6 = 0$	$7 - 1 - 6 = 0$
4-9	$15 - 5 - 5 = 5$	$10 - 5 - 5 = 0$
5-6	$16 - 7 - 4 = 5$	$11 - 7 - 4 = 0$
5-7	$15 - 7 - 8 = 0$	$15 - 7 - 8 = 0$
6-8	$17 - 11 - 1 = 5$	$17 - 11 - 1 = 5$
7-8	$17 - 15 - 2 = 0$	$17 - 15 - 2 = 0$
8-10	$22 - 17 - 5 = 0$	$22 - 17 - 5 = 0$
9-10	$22 - 10 - 7 = 5$	$22 - 10 - 7 = 5$

∴ The critical path: 1-3-5-7-8-10

[longest]

activities are

∴ All remaining the non-critical path.

PERT: (Project Evaluation Review Technique)

→ project management

1) CPM

2) PERT

CPM: The activities timing are deterministic in nature.

PERT: each activity will have 3 time estimates.

1) Optimistic time

2) Most likely time

3) Pessimistic time

i) consider the following table summarizing the details of a project.

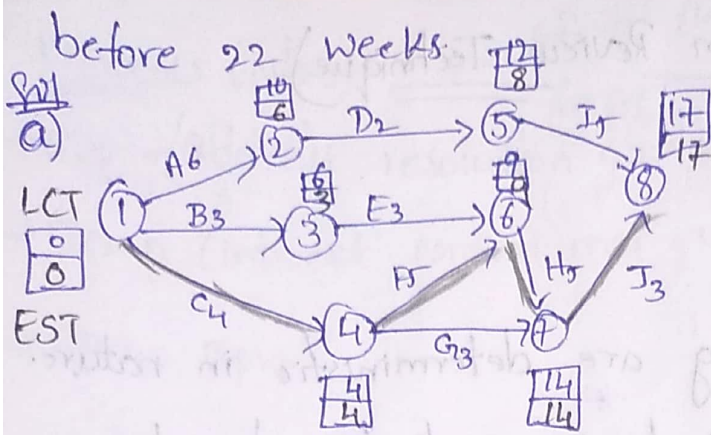
Activity	Predecessors	Duration (weeks)		
		O	M	P
A	-	5	6	7
B	-	1	3	5
C	-	1	4	7
D	A	1	2	3
E	B	1	2	9
F	C	1	5	9
G	C	2	2	8
H	EF	4	4	10
I	D	2	5	8
J	HG	2	2	8

a) construct the project N/W

b) find the expected duration & variance of each activity

c) find the critical path & expected project completion time

d) What is the ~~proba~~ probability of completing project on or



b) calculation of the expected duration & variance of each activities

Activity	duration			mean duration	Variance
	O	m	P		
A	5	6	7	6	0.11
B	1	3	5	3	0.44
C	1	4	7	4	0.11
D	1	2	3	2	0.11
E	1	2	9	3	1.78
F	1	5	9	5	1.78
G	2	2	8	3	1
H	4	4	10	5	1
I	2	5	8	5	1
J	2	2	8	3	1

Mean duration formula: $\frac{t_o + 4t_m + t_p}{6}$

$$\Rightarrow \frac{5 + 4(6) + 7}{6} = \frac{5 + 24 + 7}{6} = \frac{36}{6} = 6$$

$$\Rightarrow \frac{1 + 4(3) + 5}{6} = \frac{1 + 12 + 5}{6} = \frac{18}{6} = 3$$

∴ Critical path

⇒ 1-4-6-7-8

⇒ C-F-H-J

⇒ 4 + 5 + 5 + 3 = 17 weeks

Variance formula:

$$\sigma^2 = \left[\frac{t_p - t_o}{6} \right]^2$$

$$\Rightarrow \left[\frac{7 - 5}{6} \right]^2 = \left[\frac{2}{6} \right]^2 = 0.11$$

$$\Rightarrow \left[\frac{5 - 1}{6} \right]^2 = \left[\frac{4}{6} \right]^2 = 0.44$$

© Earliest starting time $\Rightarrow$ forward

node 1 = 0

2 = 0 + 6 = 6

3 = 0 + 3 = 3

4 = 0 + 4 = 4

5 = 6 + 2 = 8

6 = 3 + 3 = 6 $\left\{ \begin{array}{l} q_{max} \\ 4 + 5 = 9 \end{array} \right.$

7 = 4 + 3 = 7 $\left\{ \begin{array}{l} max = 14 \\ 9 + 5 = 14 \end{array} \right.$

8 = 8 + 5 = 13 $\left\{ \begin{array}{l} 14 + 3 = 17 \end{array} \right.$ (17) max

Latest completion time $\Rightarrow$ Backward

node 8 = 17

node 7 = 17 - 3 = 14

6 = 14 - 5 = 9

5 = 17 - 5 = 12

4 = 9 - 5 = 4 $\left\{ \begin{array}{l} 14 - 3 = 11 \end{array} \right.$ 4 min

3 = 9 - 3 = 6

2 = 12 - 2 = 10

1 = 10 - 6 = 4 $\left\{ \begin{array}{l} 6 - 3 = 3 \\ 4 - 4 = 0 \end{array} \right.$ 0 min

① What is the probability completing the project on or before 22 weeks

Activity	mean duration	Variance
C	4	1.78
F	5	1
H	5	1
J	3	1
$\mu = 17$		$\sigma^2 = 4.78$

$\therefore \sigma = \sqrt{4.78} = 2.189$

$$P(x \leq 22)$$

$$P\left[\frac{x - \mu}{\sigma}\right] \xrightarrow{\text{formula}}$$

$$= \frac{22 - 17}{2.19} = \frac{5}{2.19} = 2.28$$

$$P\left[\frac{x - \mu}{\sigma}\right] \leq \left[\frac{22 - 17}{2.18}\right]$$

$$= \frac{5}{2.18}$$

$$= 2.2936$$

$$\begin{aligned} \sigma^2 &= 4.74 \\ \sigma &= \sqrt{4.74} \\ &= 2.18 \end{aligned}$$

$$\therefore P(x \leq 22) = 0.9887$$

2) consider the data of a project summarized in table.

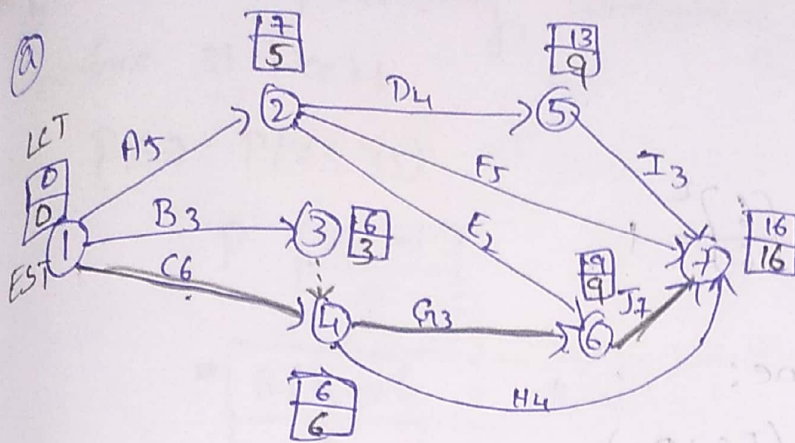
Activity	Immediate predecessor(s)	duration (weeks)		
		O(a)	m	P(b)
A	—	4	4	10
B	—	1	2	9
C	—	2	5	14
D	A	1	4	7
E	A	1	2	3
F	A	1	5	9
G	B, C	1	2	9
H	C	4	4	4
I	D	2	2	8
J	E, G	6	7	8

a) construct a project Ntw.

b) find the expected duration & variance of each activity

c) find the critical path & expected completion time.

d) what is the probability, completing the project on or before 35 weeks.



critical path

1-4-6-7

= C-G-J
= 6+3+7=16 weeks

⑥ calculating the expected duration & variance of each activity.

Activity	duration			mean duration	variance.
	O(a)	m	P(b)		
A	4	4	10	5	1
B	1	2	9	3	1.78
C	2	5	14	6	4
D	1	4	7	4	1
E	1	2	3	2	0.11
F	1	5	9	5	1.78
G	1	2	9	3	1.78
H	4	4	4	4	0
I	2	2	8	3	1
J	6	7	8	7	0.11

meanduration:

$$m = \frac{to + 4tm + tp}{6}$$

$$= \frac{4 + 4(4) + 10}{6} = \frac{4 + 16 + 10}{6} = \frac{30}{6} = 5$$

Variance

$$\sigma^2 = \left[\frac{t_p - t_o}{6} \right]^2$$

$$= \left[\frac{10 - 4}{6} \right]^2 \Rightarrow \left[\frac{6}{6} \right]^2 = 1$$

c) Earliest starting time:

$$\text{formula: } ES_j = \max_i (ES_i + D_{ij})$$

$$\text{node 2: } 0 + 5 = 5$$

$$3: 0 + 3 = 3$$

$$4: 0 + 6 = 6$$

$$5: 5 + 4 = 9$$

$$\left. \begin{array}{l} 6: 6 + 3 = 9 \\ 5 + 2 = 7 \end{array} \right\} \max: 9$$

$$7: 9 + 3 = 12$$

$$\left. \begin{array}{l} 5 + 5 = 10 \\ 9 + 7 = 16 \end{array} \right\} \max: 16$$

$$6 + 4 = 10$$

latest completion time

$$LC_j = \min_j [LC_j - D_{ij}]$$

$$\text{node: } 7 = 16$$

$$6: 16 - 7 = 9$$

$$5: 16 - 3 = 13$$

$$\left. \begin{array}{l} 4: 16 - 4 = 12 \\ 9 - 3 = 6 \end{array} \right\} 6 \text{ min}$$

$$3: 6 - 0 = 6$$

$$2: 13 - 4 = 9$$

$$\left. \begin{array}{l} 16 - 5 = 11 \\ 9 - 2 = 7 \end{array} \right\} \min 7$$

$$\text{node 1: } 7 - 5 = 2$$

$$6 - 3 = 3$$

$$6 - 6 = 0$$

d) find the probability completing the project on or before 35 weeks

$$P \leq P(x \leq 35)$$

$$P \left[\frac{x - \mu}{\sigma} \right]$$

Activity	duration	variance
C	6	4
G	3	1.78
J	7	0.11
	$\mu = 16$	5.89

$$P(x \leq 35)$$

$$P \left[\frac{x - \mu}{\sigma} \right]$$

$$\sigma^2 = 16 \Rightarrow \sigma = \sqrt{16}$$

$$= 4$$

$$\boxed{\sigma = 4}$$

$$\left[\frac{35 - 16}{4} \right]$$

$$\left[\frac{19}{4} \right]$$

$$\therefore P(x \leq 35) = 4.75$$

$$P(x \leq 35)$$

$$P \left(\frac{x - \mu}{\sigma} \right) \leq \frac{35 - 16}{2.42}$$

$$= \frac{19}{2.42}$$

$$= 7.85$$

UNIT-I

LINEAR PROGRAMMING PROBLEM

INTRODUCTION TO OPERATIONS RESEARCH

INTRODUCTION

Making a decision is becoming more and more complex. One has to take certain decisions for himself and for others. Managers are evaluated primarily on the results of their decisions and successful managers use quantitative approaches to decision making.

These approaches are also known as:

Operations Research [OR]

Management Science [MS]

Decision Science [DS]

Quantitative Techniques for Business Decision [QTBD]

Operation Research is an application of scientific approach to solve management problems in order to help managers to make better decisions. An effective decision depends on many factors, which may be economic, social and political.

Example:

Starting of a new factory at a place would depend on economic factors such as:

- a. Construction costs
- b. Labor costs
- c. Availability of raw material
- d. Transportation costs
- e. Taxes
- f. Energy costs
- g. Pollution control costs etc.,

Qualitative and Quantitative Analysis

While solving a real life problem the decision maker must examine a problem from both qualitative as well as quantitative perspective, so that data so obtained should be analyzed in both perspectives to suggest solution of problem

- ***Qualitative analysis***

It is primarily based on the judgment and experiences of the managers and includes the manager's feelings regarding the problem. Qualitative analysis is emphasized in the situations where the managers has experiences with similar problems or if the problem is relatively simple.

- ***Quantitative Analysis***

It is recommended if the problem is quite complex and the manager is not having enough experience with similar problems.

Quantitative analyses are primarily mathematical and statistical techniques are used in managerial decision making. Quantitative approach to problem solving requires collection of large amount of data and development of mathematical models. Quantitative techniques are used to support

management to make decisions under complex and uncertain situation in order to achieve the objective and goal the organization.

Example:

Consider a problem of an investor seeking advice for investments in three alternatives; stock market, real estate and bank deposit. To suggest an acceptable solution, we need to consider certain quantitative factors to be examined in the light of the problem.

ORIGIN AND GROWTH/EVOLUTION/HISTORY OF OR

However, a particular model and technique of OR was used as early as in World War-I

- In the year 1914-1915, Thomas Edison used a tactical game board to minimize shipping losses from enemy submarines instead of risking ships in actual war conditions
- About the same time, A.K Erlang developed experiments to study the fluctuations in demand for telephone facilities using automatic dialing equipment, later on such experiments were used for the development of waiting line theory.

Stage: 1: During World War-II

It is generally agreed that OR came into existence during WW-II when there was a critical need manage scarce resources in military operations. Since the war involved strategic and tactical problems which were greatly complicated and provided adequate solutions.

Therefore, groups of individuals who collectively were considered as specialists in Mathematics, Economies, Statistical and Probability theory, Engineering, Behavioral and Physical science were formed as special units within the armed forces to deal with strategic and tactical problems of various military operations.

Such groups were first formed by the British air force and later, the American armed forces formed similar groups

- At very first Britain developed radar operational research unit and was assigned to analyze the problem of radar at gun sites under the guidance of Prof. PMS Blackett which also called as “Air Battle of Britain”
- Later, United States, Canada and France started the use of OR techniques with such team of scientists.
- At the end of WW-II, the scientists of this team moved to different sectors, such as transportation, health, Education.

Stage: 2: After World War-II

After the war ended, OR approaches applied to civilian problems, business, industry, research and development etc.

There are various important factors behind the rapid development in the use of operations research approach.

- After WW-II, The economic and industrial boom resulted in continuous mechanization, automation, decentralization of operations and division of management functions and resulted in complex managerial problems. Therefore applications of OR to managerial decision-making became popular.

- In the year 1947, George B Dantzig, developed the concept of Linear Programming to allocate scarce resources by a method known as simplex method.
- At the end of 1950, many other techniques of OR developed such as SQC, Dynamic programming, queuing theory and inventory theory
- A year later, project scheduling techniques: CPM and PERT were developed as efficient tools for scheduling and monitoring lengthy, complex and expensive projects of that time.
- In 1948, an OR club was formed in England which later changed its name to the OR society of U.K. its journal, OR quarterly first appeared in 1950
- In 1952, the OR society of America (ORSA) was founded and its journal operations research was first published in 1953.
- In 1953, The Institute of management sciences [TIMS] was founded as an international society to identify, extend and unify scientific knowledge pertaining to management. Its journal, management science, first appeared in 1954.

Stage: 3: In India

In India operations research originated during the last phase of WW-II. Initially, an OR Centre was formed at the Air Head, quarterly to tackle problems relating to radar.

- In India OR started in 1949, with the opening of a unit at the regional research laboratory, Hyderabad for using OR in planning and organizing research
- In 1953, an OR unit was formed at the Indian statistical institute, Calcutta with a view to apply OR techniques to national planning under the guidance of Prof. Mahalanobis.
- OR got its formal recognition in the country with the formation of OR society of India (ORSI) in 1957. The first president of the society of P.C. Mahalanobis.
- The society became the member of international federation of OR societies in 1959 and organized its first conference at all India level during the same year.
- The publication of journal “OPSEARCH” by ORSI in 1954 further added to the growth of OR activities in India.
- An international seminar “second OR around-the-world meeting” sponsored by ORSI was held at New-Delhi during august 7-10, 1967.

Today, however, service organizations such as banks, hospitals, libraries, airlines, railways etc. recognize the usefulness of OR in improving efficiency. Because of OR’s multi-disciplinary character and application in varied fields, it has a bright future.

NATURE/CHARACTERISTICS OF OR

Operation Research can be considered as being the application of scientific method by interdisciplinary teams to problems involving the control of organized (men – machine) systems so as to provide solutions which are best for the purpose of the organization of a whole. Different characteristics constitute the nature of OR can be summed up as under:

Inter Disciplinary Team Approach

The OR has the characteristics that it is done by a team of scientists drawn from different disciplines such as mathematics, statistics, economics, engineering, physics etc., it is essentially an inter disciplinary

team approach. Each member of the OR team is benefited from the view point of the other so that a workable solution obtained through such collaborative study has a great chance of acceptance by management.

Systems Approach

OR emphasizes on the overall approach to the system. This characteristic of the OR is often referred as system oriented. The orientation is based on the observation that in the organized systems the behavior of any part ultimately has some effect on every other part. But all these effects are not significant and even not capable of detection. Therefore, the essence of system orientation lies in the systematic search for significant interactions in evaluating actions of any part of the organization.

OR can't give perfect answers or solutions to the problems it merely bad answers. To the problem which otherwise have worst answers. Thus, OR simply helps in improving the quality of the solution. But does not result into a perfect solution.

Goal Oriented Optimum Solution

OR tries to optimize a well-defined function subject to give constraints and as such as concerned with the optimization theory.

Scientific Method

OR involves scientific and systematic attack of complex problems to arrive at optimum solution. In other words, OR uses techniques of scientific research.

Thus OR comprehends both aspects i.e., it includes both scientific research on the phenomena at operating systems and the associated engineering activities aimed at applying the results of research.

Use of Models

OR use models built by quantitative measurement of the variable concerning a given problem and also derives a solution from the model using one or more of the diversified solution techniques. A solution may be extracted from a model either by conducting experiments on it or by mathematical analysis. The purpose help the management to determine its policy and actions scientifically.

DEFINITIONS OF OR

- OR is a scientific approach to problem solving for executive management
▪ **-H.M.WAGNER**
- OR may be described as a scientific approach to decision making that involves the operation of organizational system.
–**F.S.HILLER AND G.J.LIBERMAN**
- OR is a systematic application of quantitative methods, techniques and tools to the analysis of problems involving the operation of systems. **GEORGE**
- OR is essentially a collection of mathematical techniques and tools which in conjunction with a system approach, are applied to solve practical decision problems of an economic or engineering nature
-GEORGE

SCOPE/APPLICATIONS OF OR

OR is today recognized as an applied science concerned with large number of diverse human activities. To be precise on operation uses some valuable recourses like man, money, machine, time, effort etc.

An OR worker is required,

- To minimize the input value for a specific output.
- To maximize the output value for a specific input
- To maximize some functions of these values

Example: The profit (or) return on investment

Some of the areas of management where techniques of OR are applied are listed below:

In Agriculture

With the explosion of population and consequent shortage of food, quantitative techniques are very useful in sorting out problems.

- The problem of optimum allocation of land in various crops in accordance with the climatic conditions.
- The problem of optimum distribution of water from various resources, fertilizers, new techniques and capabilities.

In Finance

In this modern times of economies crisis, it has become very necessary for every government to have careful planning using the quantitative techniques applied to:

- Maximize the per capita income with minimum resources.
- Find out the profit plan for the company.
- Determine the best replacement policies.

In Industry

If the industry manager decides his politics only on the basis of his past experiences, he may not lead the organization in a positive manner and organization will suffer heavy losses. This losses can be compensated by using quantitative techniques in business management with formulates total production schedule, optimum utilization of resources available

In Marketing

With the help of quantitative techniques, a marketing manager can decide where to distribute the product for sale so that the total carriage inwards and outwards cost will be minimize.

- Minimum per unit sale price.
- Size of the stock to meet the future demand.
- Selection of advertising media with to respect to time, cost, efforts, results etc.
- Purchasing criteria at minimum possible cost

In Personnel Management

A personnel manager can use quantitative techniques

- To appoint the most suitable persons an appropriate salary.
- To determine the best age of retirement.
- To find out the number of persons to be appointed on full time basis depending upon the nature of the work load.

In Production Management

A production manager can use quantitative techniques

- To find the number and size of the items to be produced.
- In scheduling and sequencing the production run by proper allocation of machines.
- In calculating the optimum product mix.
- To select, locate and design production plans
- ***In Insurance Sector***
 - What should be the premium rates for various modes of policies?
 - How best the profits could be distributed in the case of profit generation?

LIMITATIONS OF OR

The limitations of OR are mostly related with the problems concerned with model building and the time and cost factors involved in its application rather than its practical utility. Some of these limitations are:

- OR provides solution only when the problem consists of only quantifiable elements. But in real situation, all relevant variables or influencing factors cannot be qualified and some may be qualitative in nature. Factors which cannot be quantified cannot be used in the mathematical models used in OR.
- OR tries to find out the optimal solution, considering all the factors individual in a problem. In present day business problems are affected by a number of factors and expressing them with one another requires enormous calculations. Such calculations cannot be handled manually and require electronic computers involving heavy costs. This makes application of OR suitable to only very large organizations.
- Application of mathematical model of OR requires persons having knowledge of mathematics or statistics who might not be aware of the problems related to business. On the contrary, managers who can understand business problems fail to understand the complexity of OR. this creates a gap between the two
- Application of OR becomes a costly affair when the basic data is subjected to frequent changes. Also a fairly good solution which can be implement at present (i.e., satisfying solution) may be better than an optimal solution which may be possible to implement at a late data.
- Sometimes implementation of decisions becomes a delicate task because of resistance offered by people who are affected by the decision. Hence applications of OR models should take into account complexities of human relations and behavior.

PHASES OF OR

There are three phases of the scientific model on which OR approach is based.

Judgment Phase

This phase includes

- Identification of the real life problem.
- Selection of an appropriate objective and selection of various variables related to this objective.
- Application of the appropriate sale of measurement i.e, measures of effectiveness.
- Formulation of an appropriate model of the problem.

Research Phase

This phase is the largest and longest among other two phases. This phase utilizes Observations and data collection for a better understanding of the problem.

- Formulation of hypothesis and model.
- Observation and experimentation to test the hypothesis.
- Analysis of the available information and verification of the hypothesis using pre-established measures of desirability.
- Prediction of various results from the hypothesis.
- Generation of the result and consideration of alternative method.

Action Phase

This phase consist of

- Making recommendations for implementing the decision by an individual who is in the position to implement results.
- This individual must be aware of the environment in which the problem occurred, objective, assumptions and omissions of the model of the problem.

LINEAR PROGRAMMING PROBLEM

INTRODUCTION

The term linear programming is a combination of two words, “*linear*” and “*programming*”. The term “*linear*” refers to linear relationships among the variables in a model. Thus, a given change in one variable is always cause a resulting proportional change in another variable. The word “*programming*” refers to modelling and solving problem mathematically that involves the economic allocation of limited resources by choosing a particular course of action or strategy among various alternative strategies to achieve the desire objectives.

Linear programming is a mathematical modelling technique useful for the allocation of “*source*” or “*limited*” resource, such as men (labor), machine, material, time, warehouses, capital etc., to several competing activities such as products, services, jobs, new projects on the basis of a given criterion of optimality.

Generally the criterion of optimality is return on investment, profit, cost, time, distance etc.,

In 1947, during world war-2, “George .B. Dantzing” while working with the US air force developed LP model, primarily for solving military logistics problems.

But now, it is being used extensional in all functional areas of management airlines, agriculture, military operations, education, transportation planning and research and development etc.,

STRUCTURE AND MODEL OF LPP

STRUCTURE

The genera structure of LPP consist of three basic components. They are

a: Decision Variables

The LPP can be applied to the given problem when the problem contains various alternatives [courses of actions] for arriving the optimal value of objective function. These alternatives are evaluated by nature of objective function and availability of resources. For this we study certain activities those are called “*decision variables*”. Usually denoted $x_1, x_2, \dots, x_n$.

Example:

In a product mix, contains 3 products viz., A, B and C and the number of units for product A is x_1 , B is x_2 and C is x_3 are the decision variables.

The values of these variables may or may not be under the decision makers, then such variables are said to be “*controllable*” otherwise “*uncontrollable*”

But in an LPP all decision variables are controllable continuous and non-negative i.e. $x_j \geq 0$; $j=1, 2, \dots, n$.

b: Objective Function

The objective function of each LPP is expressed in terms of decision variables to optimize [maximize or Minimize] the criterion of optimality such as profit, cost, revenue, distance etc.

It is represented as:

$$\begin{aligned}\text{Optimize [Maximize or Minimize]} Z &= \sum_{j=1}^n C_j X_j \\ &= c_1 x_1 + c_2 x_2 + \dots + c_n x_n\end{aligned}$$

Where Z = The optimality, which is a function of decision variables x_j ; $j=1,2,\dots$

C_j = Parameters that represents the contribution of a unit of the respective variables $x_1, x_2, \dots, x_n$

Note of Book:

- The optimal value of the given objective function is obtained by the graphical or simplex method.

c: Constraints

In every organization certain limitations or constraints are available on the use of resources viz., men, machine, material, capital etc. that limit the degree to which an objective can be achieved.

Such constraints must be expressed as linear equalities or inequalities in terms of decision variables.

Example:

Consider the labor constraint so as to use limited labor (50) for 3 products. For 3 products the decision variables are considered as x_1, x_2, x_3 and labor constraint is as: $x_1 + x_2 + x_3 \leq 50$

Note of Book:

- The solution of LPP must satisfy these constraints.

Mathematical Model of LPP

The general mathematical model of LPP contains 3 components as follows.

Objective Function:

$$\begin{aligned}\text{Optimize [Maximize or Minimize]} Z &= \sum_{j=1}^n C_j X_j \\ &= c_1 x_1 + c_2 x_2 + \dots + c_n x_n\end{aligned}$$

$$\begin{aligned}\text{STC:} \quad & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \quad [\leq, =, \geq] b_1 \\ & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \quad [\leq, =, \geq] b_2 \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \quad [\leq, =, \geq] b_i \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \quad [\leq, =, \geq] b_m\end{aligned}$$

Restrictions: $x_j \geq 0$; $j=1,2,\dots,n$

Note of Book:

- In any type of LPP, there will be 'n' number of decision variables and 'm' constraints.
- In LPP, the expression [$\leq$, $=$, $\geq$] means that in any specification problem each constraint may take only one of the three possible forms.
 1. $\leq$: Less than or equal to
 2. $\geq$: Greater than or equal to
 3. $=$: Is equals to

FORMULATION OF LPP

The following steps to be followed while formulating an LP Problem:

Step: 1: Identify the decision variables

Firstly, identify number of products 'n' and let the decision variables [$x_j \geq 0$; $j=1,2,\dots,n$] for n products.

Example:

A, B are two products in the given problem.

Let x_1 and x_2 are number of units of products A and B

Step: 2: identify the problem data

Identify the contribution [co-efficient] of decision variables for both objective function [C_j] and constraints [a_{ij}]

Step: 3: Formulate the objective function

Identify whether the objective function is to be maximized or minimized. Then formulate the objective function into a linear mathematical expression in terms of decision variables multiplied by their profit or cost contribution [co-efficient of decision variables, C_j]

$$\text{Maximize or Minimize } Z = \sum_{j=1}^n C_j X_j = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

Step: 4: Formulate the constraints

Express the constraints verbally in terms of requirements, availability and equals to each resource.

The constraints are expressed by the resource availability as linear equality or inequality in terms of decision variables [x_j] multiplied by their profit or cost contribution [C_j] as follows:

- a. *At least as large as [minimum requirements]*
In this case the STC are expressed as greater than or equals to [$\geq$] and is used when the objective function is minimization.
- b. *No larger than [maximum availability]*

In this case the STC are expressed as less than or equals to [$\leq$] and is used when the objective function is maximization.

- c. *Exactly equals to*
- d. In this case the STC are expressed as equality [=] and is used when the objective function is both maximization and minimization.

Step: 5: Formulate LPP

Finally, formulate LPP as follows

Objective function: Maximize Z or Minimize $Z = \sum_{j=1}^n C_j X_j$

STC: $\sum_{j=1}^n a_{ij} x_j [\leq \text{ or } = \text{ or } \geq] b_i ; i=1,2,\dots,m$

Restrictions: $x_j \geq 0; j=1,2,\dots,n$

Assumptions of LPP

- The basic requirements of LPP is that both the objective and constraints must be expressed in terms of linear equalities and inequalities.
- The parameters and contribution must be known and fixed.
- The values of decision variables [x_j] must be non-negative i.e. $x_j \geq 0; j=1,2,\dots,n$

SOLUTION OF LPP

Linear programming problem can be solved by using any one of the following method.

1. Graphical Method
 - a. Maximization
 - b. Minimization
 - c. Mixed constraints
 - d. Unbounded solution
 - e. Infeasible solution
2. Simplex Method
 - a. Simple simplex method
 - b. Two-Phase method
 - c. Big-M method
 - d. Dual-Simplex method

Graphical Method

For LP Problems that have only two variables it is possible that the entire set of feasible solutions can be displayed graphically by plotting linear constraints on a graph paper to locate the best (optimal) solution. The technique used to identify the optimal solution is called the “*graphical solution approach or technique*” for an LP Problem with two variables i.e. x_1 and x_2 .

Important definitions

Solution: The set of values of decision variables $x_j(j=1,2,\dots,n)$ which satisfy the constraints of an LP Problem is said to constitute *solution* to that LP Problem.

Feasible Solution: The set of values of decision variables $x_j(j=1,2,\dots,n)$ which satisfy all the constraints and non-negativity conditions of an LP Problem simultaneously is said to constitute the *feasible solution* to that LP Problem.

Infeasible Solution: The set of values of decision variables $x_j(j=1,2,\dots,n)$ which do not satisfy the constraints and non-negativity conditions of an LP Problem simultaneously is said to constitute the *infeasible solution* to that LP Problem.

Basic Solution: For a set of 'm' simultaneous equations in 'n' variables ($n > m$), a solution obtained by setting (n-m) variables equal to zero and solving for remaining 'm' equations in 'm' variables is called a *basic solution*.

The (n-m) variables whose value did not appear in this solution are called non-basic variables and the remaining 'm' variables are called basic variables.

Basic Feasible Solution: A feasible solution to an LP Problem which is also the basic feasible solution is called the *basic feasible solution*. That is, all basic variables assume non-negative values. Basic feasible solutions are of two types:

- a. **Degenerate:** A basic feasible solution is called *degenerate* if value of at least one basic variable is zero.
- b. **Non-degenerate:** A basic feasible solution is called *non-degenerate* if values of all 'm' basic variables are non-zero and positive.

Optimal Basic Feasible Solution (OBFS): A basic feasible solution which optimizes (Maximization or Minimization) the objective function value of the given LP Problem is called *OBFS*.

Unbound Solution: A solution which can increase or decrease the value of objective function of the LP Problem indefinitely is called an *unbounded solution*.

GRAPHICAL ALGORITHM

The following steps are to be followed while using graphical method.

Step: 1: Formulation of LPP

Formulate the LP Problem in mathematically as follows:

Maximize Z or Minimize $Z = \sum_{j=1}^n C_j X_j$

STC: $\sum_{j=1}^n a_{ij} x_j [\leq \text{ or } = \text{ or } \geq] b_i ; i=1,2,\dots,m$

Restrictions: $x_j \geq 0; j=1,2,\dots,n$

Step: 2: Construction of Graph

- a. Convert the inequality constraints into equalities which satisfies non-negative variables i.e. $x_1, x_2 \geq 0$.
- b. Determine the points of each given constraint. Randomly assign '0' value to x_1 and determine x_2 value and assign '0' value to x_2 and determine x_1 value for each given constraint.
- c. Now construct the graph by drawing horizontal and vertical lines which represented by X_1 -axis and X_2 -axis in the criterion X_1OX_2 , since all the points which satisfies the constraints must in the first quadrant only.
- d. Now plot these sets of values on the graph, connect these points by a straight line. Thus, there will be as many straight lines as there are equations, each straight line represents one constraint.

Step: 3: Identification of Feasible Solution

- a. For less than ($\leq$) constraints, the feasible area will be the area which lies below the straight lines towards the origin in the graph.
- b. For greater than ($\geq$) constraints, the feasible area will be the area which lies above the straight lines opposite to the origin in the graph.
- c. Finally, the feasible area will be the common area which satisfies all the constraints and restrictions.

Step: 4: Identification and determination of Extreme points

Identify the extreme or corner points of feasible area and determine the points using given equations.

Step: 5: Evaluation of Objective Function

Evaluate the objective function at various extreme points to get alternative solutions.

Step: 6: Determination of Optimal Value

The optimal value (maximization or Minimization) of objective function occurs at a particular point.

Simplex Method

The concept of simplex method is similar to the graphical method. In graphical method, extreme points of the feasible solution space are examined to search for optimal solution at one of them.

For LP Problem with several variable ($n > 0$) i.e. $x_1, x_2, \dots, x_n$, we may not be able to graph the feasible region. Thus, we need a more efficient method to suggest an optimal solution for such problems, called the "*simplex method*". The method was developed by G B Dantzing in 1947.

The simplex is an important term in mathematics that represents an object in n -dimensional space connecting $(n+1)$ points. In one dimension, a simplex is a line segment connecting 2 points; in two dimensions, it is a triangle formed by joining 3 points; in three dimensions, it is a four sided pyramid having 4 points.

The simplex method examines the extreme points in a systematic manner, repeating the same set of steps of the algorithm until an optimal solution is reached. It is for this reason that it is also called the “*iterative method*”.

Mathematical form of an LP Problem

The standard form of the LP Problem is represented as follows:

$$\begin{aligned}\text{Optimize [Maximize or Minimize]} Z &= \sum_{j=1}^n C_j X_j \\ &= c_1 x_1 + c_2 x_2 + \dots + c_n x_n + 0s_1 + 0s_2 + \dots + 0s_n\end{aligned}$$

$$\begin{aligned}\text{STC:} \quad & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + s_1 = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + s_2 = b_2 \\ & \cdot \quad \quad \cdot \quad \quad \cdot \quad \quad \cdot \quad \quad \cdot \\ & a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n + s_i = b_i \\ & \cdot \quad \quad \cdot \quad \quad \cdot \quad \quad \cdot \quad \quad \cdot \\ & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + s_m = b_m\end{aligned}$$

Restrictions: $x_j \geq 0; j=1,2,\dots,n$

Note of Book:

- The objective function should be of either the *maximization* or *minimization* type.
- The right hand side of each constraint should be made non-negative; if it is not, this should be done by multiplying both sides of the resulting constraint by ‘-1’.
- All the constraints should be expressed as equations by adding additional variables.

Additional Variables

Additional variables are added in the given LP Problem to convert it into the standard form for the following reasons:

- These variables allow us to convert inequalities into equalities, there by converting the given LP Problem into a form that is amenable to algebraic solution.
- These variables permit us to make a more comprehensive economic interpretation of a final solution.
- Help us to get an initial feasible solution represented by the columns of the identity matrix.

The three types of additional variables, viz.,

- a. *Slack variables(s)*: A slack variable represents unused resource, either in the form of time on a machine, labor hours, money, warehouse space or any number of such resources in various business problems. Since these variables yield on profit, there such variables are added to the original object function with zero coefficients.

- b. *Surplus variables(s)*: A surplus variable represents amount by which solution values exceed a resource. These variables are also called negative slack variables. Surplus variables, like slack variables carry a zero coefficient in the objective function.
- c. *Artificial variables(A)*: Artificial variables have no meaning in a physical sense and are only used as a tool for generating an initial LP Problem solution. Artificial variables are added to the original objective function with ‘-M’ coefficients [Max Z] and ‘+M coefficients [Min Z]

Simple Simplex Method [maximization]

Simple simplex method is used to solve the LP Problem with maximization objective. The steps of the simplex method to obtain an optimal solution (if it exists) to an LP Problem are as follows:

SIMPLEX ALGORITHM

Step: 1: Formulation of LPP

To formulate the LPP follow the below given rules:

- Formulate the mathematical model of the given LPP.
- Check whether all the b_i [$i=1,2,\dots,m$] values are positive i.e. $b_i > 0$.
- Add additional variables to the given each constraint to convert into equalities.

(or)

Convert the inequality constraints into equalities by adding additional variables to the left hand side of each constraint and assign coefficients to those in objective function.

Note of Book:

- The summary of the additional variables to be added in the given LP Problem to convert it into a standard form is given in the following table.

Sl. No	Type of constraint	Type of variable	Type of objective function	
			Maximize Z	Minimize Z
1	Less than or equal to ($\leq$)	A variable is added	0	0
2	Greater than or equal to ($\geq$)	A surplus variable is subtracted	0	0
		An artificial variable is added	-M	+M
3	Equals to ($=$)	An artificial variable is added	-M	+M

- Before getting the final simplex solution, all artificial variables must be dropped out from the table.
- Consider all additional variables as basic variables.

Step: 2: Setup the initial solution

Write down the coefficients of all the variables in the LP model in the tabular form as shown below:

Initial simplex table

C_B	B	C_j	C_1	C_2	C_3		C_n	0	0	0.....0
		$b_i = X_B$	X_1	X_2	X_3		X_n	S_1	S_2	S_3 S_m
$(0)C_{B1}$	S_1	b_1	a_{11}	a_{12}	a_{13}		a_{1n}	1	0	0.....0
$(0)C_{B2}$	S_2	b_2	a_{21}	a_{22}	a_{23}		a_{2n}	0	1	0.....0
.	.	.	.	.	.		.	.	.	
.	.	.	.	.	.		.	.	.	
.	.	.	.	.	.		.	.	.	
$(0)C_{Bm}$	S_m	b_m	a_{m1}	a_{m2}	a_{m3}		0	0	0	1
$Z_j = \sum C_{Bi} X_j$			0	0	0		0	0	0	0
$C_j - Z_j$			C_1	C_2	C_3		C_n	0	0	0.....0

Where,

C_B = Coefficient of Basic variables

B = Basic variable

b_1 = value of basic variable when $X_j = 0$

C_{j2} = Incoming unit (profit or cost) or coefficient of decision variables in objective function

Z_j = Outgoing unit (profit or cost)

$C_j - Z_j$ = Net profit or cost

Step: 3: Test for optimality

Calculate the $(C_j - Z_j)$ value for all non-basic variables. Examine the values of $(C_j - Z_j)$ as follows:

- If all $(C_j - Z_j) \leq 0$, then the basic feasible solution is optimal.
- If at least one $(C_j - Z_j) > 0$, then it indicates that an improvement in the value of objective function Z is possible.

Step: 4: Test for feasibility (variable to leave the basis)

Select an entering variable to leave the basic variable by using the following steps:

- Select a column which has largest positive $(C_j - Z_j)$ value and called as key column.
- Calculate minimum ratio for each row.

$$MR = \frac{b}{kce} = \frac{b}{\text{key column element}}$$

- c. Select a row which has minimum positive value of MR and called as key row.
- d. Select an intersect element of key column and key row and called as key element.
- e. Select the leaving element [basic variable] in key row and replace it by an entering element [decision variable] from key column.

Step: 5: Finding the new solution

The new solution can be calculated based on rows and the new values can be calculated for 'b' and X_j , and s_i columns as follows:

Key row changes:

- a. If the key element is '1', then the key row remains the same in the new simplex table.
- b. If the key element is other than 1, then divide each key row element by the key element, i.e.

$$\text{Key row changes} = \frac{\text{key row element}}{\text{key element}}$$

Other row changes (ORC):

- a. $\text{ORC} = (\text{old row element in that row}) - (\text{corresponding Key row element} * \text{Fixed Ratio})$
 $\text{Fixed Ratio} = \frac{\text{key column element}}{\text{key element}}$
- b. Formulate new simplex table by using new values

Step: 6: Repeat the procedure

Go to step: 3 and repeat the procedure until all $(C_j - Z_j)$ values are either negative or zero i.e. $C_j - Z_j \leq 0$

UNIT-II

TRANSPORTATION PROGRAMMING PROBLEM

INTRODUCTION

The transportation problem deal with the distribution of a product manufactured at ‘m’ number of plants (or) factories (supply origins) to ‘n’ number of warehouses (or) retailers (demand destinations).

There are various types of transportation models and the simplest of them was first presented by F L Hitchcock in the year 1941. It was further developed by T C koomans in the year 1949 and G B Dantzing in the year 1951. Several extensions of transportation model and methods have been subsequently developed.

Meaning

- “A shipment of commodities takes place from source to destination directly is called *transportation*.”
- “*The Transportation problem is to transport various amounts of a single homogeneous commodity that are initially sored at various origins, to different destinations in such a way that the total transportation cost is a minimum*”.

Example:

Three factories F1, F2 and F3 are producing the same product. From these factories, the product is distributed to three warehouses W1, W2 and W3. Each factory has a limited supply and each warehouse has specific demand. Each factory can transport to each warehouse but the transportation cost vary from different combination. The problem is to determine the quantity each factory should transport to each warehouse in order to minimum the total transportation cost.

MATHEMATICAL MODEL OF TPP

Let there be ‘m’ number of sources S1, S2,.....,Sm having a_i ($i=1,2,.....m$) units of supply (or capacity) respectively, to be transported among ‘n’ number of destinations D1, D2,.....,Dn with b_j ($j=1,2,.....n$) units of demand (or requirement) respectively.

Let ‘ c_{ij} ’ be the cost of shipping one unit of commodity from source ‘i’ to destination ‘j’ for each route. If ‘ x_{ij} ’ represents number of units shipped per route from source ‘i’ to destination ‘j’, the problem is to determine the transportation schedule so as to minimize the total transportation cost satisfying supply and demand conditions. Mathematically the transportation problem may be stated as follows:

Objective function: Minimize $Z = \sum_{i=1}^m \sum_{j=1}^n C_{ij} X_{ij}$

STC:

Supply constraint $\sum_{j=1}^n x_{ij} = a_i ; i=1,2,.....n$

Demand constraint $\sum_{i=1}^m x_{ij} = b_j ; j=1,2,.....m$

Restrictions: $x_{ij} > 0$ (non-negative variables)

Matrix form of TPP

		Destinations	Supply
		$D_1 D_2 D_3 \dots D_n$	
Sources	S_1	$c_{11} c_{12} c_{13} \dots c_{1n}$	a_1
	S_2	$c_{21} c_{22} c_{23} \dots c_{2n}$	a_2
	.		.
	.		.
	S_m	$c_{m1} c_{m2} c_{m3} \dots c_{mn}$	a_m
Demand		$b_1 b_2 b_3 \dots b_n$	$\sum a_i = \sum b_j$

SOLUTION OF TPP

The transportation problem can be solved in two stages as follows:

1. Initial Basic Feasible Solution (IBFS)
2. Optimal Basic Feasible Solution (OBFS)(or)Test for optimality

Note of Book:

- The following algorithm is used to obtain the solution of TPP.

TRANSPORTATION ALGORITHM**Step: 1: Formulation of TPP**

Formulate the problem and arrange the data in the matrix form ($m \times n$), where m = number of rows and n = number of columns.

Step: 2: Obtain an Initial basic Feasible Solution (IBFS)

The IBFS solution is obtained by any one the following method.

- a. North West Corner Method (NWCN)
- b. Least Cost Method (LCM)
- c. Vogel's Approximation Method (VAM) or Penalty Method

Step: 3: Test for optimality

Test whether the IBFS is optimal or not by using Modified Distribution (MODI) method.

Step: 4: Repeat the procedure

Go to step:3 and repeat the procedure until the optimal solution is reached.

Note of Book:

- Check the supply and demand and both should be same i.e. $\sum S_i = \sum D_j$, so that the problem is said to be as *balanced transportation problem*.
- If $\sum S_i \neq \sum D_j$, the problem is said to *unbalanced transportation problem*.

- The number of positive allocations must be equals to $(m+n-1)$, where 'm' is number of rows and 'n' is number of columns
 - The problem with $(m+n-1)$ allocations is said to *non-degeneracy transportation problem*.
 - The problem with less than $(m+n-1)$ allocations is said to *degeneracy transportation problem*.
- Cells in the transportation table having positive allocation will be called occupied cells, otherwise empty or non-occupied cells.

Initial Basic Feasible Solution (IBFS)

The IBFS solution is obtained by any one the following method.

- a. North West Corner Method (NWCM)
- b. Least Cost Method (LCM)
- c. Vogel's Approximation Method (VAM) or Penalty Method

North West Corner Method (NWCM) Algorithm

Step: 1: Select North-West corner cell of the given transportation table and allocate as many units as possible available supply and demand requirement.

Step: 2: If for any cell, supply equals to demand, then the next allocation can be made in cell either in the next row or column.

Step: 3: Continue the procedure until the total available quantity is fully allocated to the cells as required.

Step: 4: Determine the total transportation cost using $\sum C_{ij}X_{ij}$.

Least Cost Method (LCM)

Step: 1: Select the lowest transportation cost cell (minimum positive) among all the rows or columns of the transportation table and allocate as many units as possible available supply and demand requirement.

Step: 2: If the cells are tie, select the least cost cell, which has maximum possible from supply or demand

Step: 3: Repeat the procedure until the optimal solution is reached.

Step: 4: Determine the total transportation cost using $\sum C_{ij}X_{ij}$.

Vogel's Approximation Method (VAM) or Penalty Method

Step: 1: Calculate penalties for each row and column by taking the difference between least cost cell and next least cost cell of the given transportation table.

Step: 2: Identify the row or column which has maximum penalty and select least cost cell of the row or column and allocate as many units as possible available supply and demand requirement.

Step: 3: If there is a tie occurs in the value of penalty, select the least cost cell of that row or column. If there is a tie in the least cost cell also, then select the cost cell of row or column which with have maximum possible assignment.

Note of Book:

- If there is any row or column with zero supply or demand, then it should not use in calculating future penalties.

Step: 4: Repeat the above procedure until the optimal solution is reached.

Step: 5: Determine the total transportation cost using $\sum C_{ij}X_{ij}$.

Note of Book:

- The total cost relationship among NWCM, LCM and VAM is written as follows:
$$\text{NWCM} \geq \text{LCM} \geq \text{VAM}$$

Optimal Basic Feasible Solution (OBFS)

OBFS of transportation problem can be determined by using MODI method.

Modified Distribution Method

This method is best method and time saving over the stepping stone method for solving transportation problems. The following algorithm is used to test of optimality.

MODI ALGORITHM

Step: 1: Determination of IBFS

Determine the IBFS by using VAM (Penalty method)

Step: 2: calculation of U_i and V_j values

Calculate U_i and V_j values for each row and column by using occupied cells only. i.e. $C_{ij}=U_i+V_j$

Note of Book:

- Consider $U_i=0$ or $V_j=0$, where the maximum allocated cells at a row or column.
- If the equal number of allocations are available, select row or column which has maximum number of supply or demand.

Step: 3: Obtain opportunity cost

Determine the opportunity cost for unoccupied cells as follows:

$$d_{ij}=C_{ij} - U_i - V_j$$

Step: 4: Test for optimality

If the opportunity cost (d_{ij}) is greater than or equals to zero, then the solution is said to be an optimal solution i.e. $d_{ij} \geq 0$.

Note of Book:

- If one or more unoccupied cells have negative opportunity cost i.e. $d_{ij} < 0$, the solution is not optimal and can be improved.

Step: 5: Develop new solution

If the negative values are more than one cell, select the minimum negative value of unoccupied cell and make a star (*).

Step: 6: Draw a closed path or loop

Draw a closed loop or path from unoccupied (selected*) cell in clockwise direction and permitted only for occupied cells and ends with original unoccupied (selected*) cell.

Note of Book:

- Every closed path or loop must have even number of turns and it formed with horizontal and vertical lines only.

Step: 7: Develop new opportunity table

- Assign alternative signs positive (+) and negative (-) signs at the corner points of the closed path starting with a positive sign to the selected (*) cell.
- Select the negative sign cell which contains minimum allocation of units and then add that value to all positive sign cells and subtract that value from the negative sign cells.
- Determine the total transportation cost using $\sum C_{ij}X_{ij}$.

Step: 8: Repeat the procedure

Go to step:2, repeat the procedure until the optimal solution is reached.

UNBALANCED TPP

“If the sum of supply is not equal to sum of demand in the given matrix, then the problem is called unbalanced transportation problem”

Note of Book:

- If the supply is less than demand, then add the dummy row with zero (0) unit cost values.
- If the demand is less than supply, then add the dummy column with zero (0) unit cost values.
- Apply VAM to obtain IBFS.

ASSIGNMENT PROGRAMMING PROBLEM

INTRODUCTION

Assignment problem is used, when a particular job or machine is to be assigned to a particular worker based on his training and proficiency level or a project manager has to deputed to a particular project based on his qualification, experience and special exposure to that special type of project.

Meaning

“An Assignment Problem is to assign ‘n’ number of origins (jobs) to the equal ‘n’ number of destinations (persons) at a minimum cost (or maximum profit)”.

Thus, the problem is how the assignments should be made so as to optimize the given objective.

Applications

Some of the problems where the assignment technique may useful are:

- a. Assignment of workers to machines.
- b. Assignment of salesmen to different sales areas.
- c. Assignment of clerks to various checkout counters.
- d. Assignment of classes to rooms.
- e. Assignment of vehicles to routes.
- f. Assignment of contracts to bidders.

MATHEMATICAL MODEL OF APP

Let there be ‘n’ number of sources of facilities or workers $W_1, W_2, \dots, W_n$ having a_i ($i=1,2,\dots,n$) units of supply (or capacity) respectively, to be assigned among ‘n’ number of activities or jobs $J_1, J_2, \dots, J_n$ with b_j ($j=1,2,\dots,n$) units of demand (or requirement) respectively.

Let ‘ c_{ij} ’ be the effectiveness (cost, profit, time etc.,) of assigning each facility to each job to determine the assignment schedule so as to optimize (maximize profit or minimize cost or time) the given objective. Mathematically the assignment problem may be stated as follows:

Objective function: Maximize or Minimize $Z = \sum_{i=1}^m \sum_{j=1}^n C_{ij} X_{ij}$

STC:

Supply constraint $\sum_{j=1}^n x_{ij} = a_i = 1; i=1,2,\dots,n$

Demand constraint $\sum_{i=1}^m x_{ij} = b_j = 1; j=1,2,\dots,m$

Restrictions: $x_{ij} = 1$ (non-negative variables)

Assignment problem as special case of transportation problem

The assignment problem is seen to be the special case of transportation problem when each origin is associated with one and only one destination. In such case, $m=n$ and the numerical evaluations of such association are called ‘effectiveness’ instead of ‘transportation’. Mathematically, all a_i and b_j are unity (1), and each x_{ij} is limited to ‘1’.

Matrix form of APP

		Activities (jobs)	Supply
		$J_1 J_2 J_3 \dots J_n$	
Facilities or Workers	W_1	$c_{11} c_{12} c_{13} \dots c_{1n}$	1
	W_2	$c_{21} c_{22} c_{23} \dots c_{2n}$	1
	.		.
	.		.
	.		.
	W_n	$c_{n1} c_{n2} c_{n3} \dots c_{nn}$	1
Demand		1 1 1.....1	$\sum a_i = \sum b_j = n$

SOLUTION OF APP

Assignment problem can be solved by using any one of the following method.

1. Hungarian Method
 - a. Minimization
 - b. Maximization
 - c. Unbalanced Assignment problem
 - d. Travelling salesman method

Hungarian Method

This method is used to solve assignment problem so as to obtain optimal assignment schedule.

HUNGARIAN ALGORITHM**Step: 1: Formulation of APP**

Formulate the problem and arrange the data in the matrix form ($n \times n$), where n = number of rows = number of columns.

Step: 2: Determination of the Opportunity table

- a. Row operations: Locate the smallest element in each row of the given table and subtract that element from each element of that row.
- b. Column operations: Locate the smallest element in each column of the modified (row operations) table and subtract that element from each element of that column.

Note of Book:

- Each row and column must have at least one zero (0) value.

Step: 3: Making an Assignment

- a. Identify rows successively from top to bottom until a row with exactly one zero element is found.
- b. Make an assignment to single zero by making '0' box ☐
- c. For each zero value that becomes assigned, eliminate all other zeros in the same rows and columns by crossing the zero [X].
- d. Apply the above steps for each column also.
- e. Continue this procedure until all zeros in the rows and columns are either assigned or crossed off.

Step: 4: Test for optimality

If the number of assigned cells equals to the number of rows or columns then it is said to be optimal solution.

Hence the total value can be calculated by adding the original values of assigned cells from the given original table.

Note of Book:

- If the number of assignments are less than to number of rows or columns, then the solution is not optimal and can be improved.

Step: 5: Revise the opportunity table

Draw a set of horizontal and vertical lines to cover all the zeros in the revised table as follows:

- a. For each row in which no assignment was made, make a tick mark [$\sqrt{}$].
- b. Evaluate the marked rows, if any zero cell occurs in these rows, make a tick mark [$\sqrt{}$] to the respective columns that contains zeros.
- c. Evaluate the marked columns, if any assigned zeros occurs in those columns, make a tick mark [$\sqrt{}$] to the respective rows that contains assigned zeros.
- d. Now draw straight lines through each marked column and such unmarked row.

Note of Book:

- If the number of lines drawn is equal to the number of rows or columns, the current solution is optimal.

Step: 6: Develop the new revised opportunity table

- a. Select the smallest element from the cells those not covered by any line and assumed as 'k'.
- b. Subtract that element 'k' from every element in the table those not covered by any line.
- c. Add that element to every element in the table those covered by the two lines i.e., intersection of two lines.
- d. Elements in the table covered by only one line remains same (unchanged)

Step: 7: Repeat the procedure

Go to step: 3, repeat the procedure until the optimal solution is reached.

Note of Book:

- Check the number of rows and number of columns both should be same, so that the problem is said to be as *balanced assignment problem*.
- If the number of rows is not equal to the number of columns, the problem is said to *unbalanced assignment problem*.
- The number of positive assignments must be equals to 'n', where $n = \text{number of rows} = \text{number of columns}$
- Cells in the assignment table having positive assignments will be called assigned cells, otherwise empty or non-assigned cells.

Minimization

This method is applied to obtain minimum cost or time required to complete a job.

Maximization

This method is applied to obtain maximum profit after completion of a job. The following steps are used to solve the maximum assignment problem.

Step: 1: Conversion of problem

Convert the given maximization problem into minimization problem by subtracting all cells from the maximum value (cell).

Step: 2: Apply Hungarian method to the minimization problem

Note of Book:

- Calculate total value by adding the original values from the given table.

Unbalanced Assignment problem

“If the number of rows is not equal to the number of columns in the given matrix, then the problem is called unbalanced assignment problem”

Note of Book:

- If the number of rows is less than number of columns, then add a row with zero unit values.
- If the number of columns is less than number of rows, then add a column with zero unit values.
- Apply Hungarian method to obtain optimal value.

Travelling Salesman Method

In this method a travelling salesman travel starting from his home city [the city from where he started], visiting each city only once and returning to his home city so that the total distance or cost or time is minimum.

Note of Book:

- There is no assignment for the same cities like; A-A, B-B etc.,

Mathematical model of travelling salesman method

		To Cities	Supply
		A BC.....n	
Form City	A	- $c_{12} c_{13} \dots c_{1n}$	1
	B	$c_{21} - c_{23} \dots c_{2n}$	1
	.		.
	.		.
	.		.
	n	$c_{n1} c_{n2} c_{n3} \dots -$	1
Demand		1 1 1.....1	$\sum a_i = \sum b_j = n$

Note of Book:

- To solve the TSM, apply Hungarian method.

- If the sequence [Eg: A-B-C-D-E-A] is not reached, consider the least positive (>0) value from the cells as zero (0) and make an assignment.
- Repeat the above step until the sequence is reached.

GATE

UNIT-IV

GAME THEORY

INTRODUCTION

The term game refers to a situation of conflict and competition in which two or more competitors (or participants) are involved in decision-making in anticipation of certain outcomes over a period of time. The competitors are referred as players. A player may be an individual, a group of individuals or an organization.

Example:

- Two player game- chess.
- Pricing of products, where a firm's ultimate sales are determined not only by the price levels it selects but also by the prices its competitors set.
- Success of an advertising/marketing campaign depends largely on various types of services offered to the customers, etc.

As an area of academic study, theory of games provides a series of mathematical models that may be useful in explaining interactive decision-making concepts where two or more competitors are involved under conditions of conflict and competition.

Game theory came into existence in 20th century. However, in 1944 John Von Neumann and Oscar Morgenstern published a book "Theory of games and economic behaviour", where they discussed how businesses of all types may use this technique to determine the best strategies given a competitive business environment. Authors' approach was based on the principle of "best out of the worst".

Important definitions

Number of players: If a game involves only two players [competitors], then it is called a two-person game. However, if the number of players is more, the game is referred to as n-person game.

Gain, Loss and Value of the Game: The positive result of one player is gain and the same is considered as loss to another player.

If in a game sum of the gains to one player is exactly equal to the sum of losses to another player, so that sum of the games and losses equals to zero, then the game is said to be a *zero-sum game*. Otherwise it is said to be a *non-zero sum game*.

Payoff matrix: The payoffs in terms of gains or losses, when players select their particular strategies, can be represented in the form of a matrix, called the payoff matrix.

The rows of the payoff matrix denote player A's strategies and the columns denote player B's strategies.

Since player A is assumed to be the gainer always, so he wishes to gain as large a payoff a_{ij} as possible while player B [looser] will do his best to reach as small a value of a_{ij} as possible. If player B is gainer then player will be looser.

Player A has 'm' strategies represented by the letters: $A_1, A_2, \dots, A_m$ and player B has 'n' strategies represented by the letters: $B_1, B_2, \dots, B_n$. The number m and n need not be equal. The total

number of possible outcomes is therefore $m \times n$. Here, it is assumed that each player knows not only his own list of possible courses of action but also of his opponent.

		Courses of action (Strategies)							Probability
		S_1	S_2	S_3		S_j		S_n	
States of nature	N_1	P_{11}	P_{12}	P_{13}		P_{1j}		P_{1n}	P_1
	N_2	P_{21}	P_{22}	P_{23}		P_{2j}		P_{2n}	P_2
	.								.
	.								.
	N_i	P_{31}	P_{32}	P_{33}		P_{3j}		P_{3n}	P_i
	.								.
	.								.
	N_m	P_{m1}	P_{m2}	P_{m3}		P_{mj}		P_{mn}	P_m

TWO-PERSON ZERO-SUM GAMES

A game with only two players, say player A and player B is called a *two-person zero-sum game*, if one player's gain is equal to the loss of other player so that total sum is zero.

Since the game is zero-sum, the gain of one player is equal to the loss of other and vice versa.

STRATEGY: The strategy for a player is the list of all possible actions [courses of action] that he will take for every payoff [outcome] that might arise.

- Player A's strategies, $S(A) = \{ p_1, p_2, \dots, p_m \}$
- Player B's strategies, $S(B) = \{ q_1, q_2, \dots, q_m \}$

Optimal Strategy

The particular strategy by which a player optimises his gains or losses without knowing the competitors strategies is called optimal strategy.

Pure Strategy

It is the decision rule which is always used by the player to select the particular strategy [course of action]. Thus, each player knows in advance of all strategies out of which he always selects only one particular strategy regardless of the other player's strategy, and the objective of the players is to maximize gains or minimize losses.

Mixed Strategy

Courses of action that are to be selected on a particular occasion with some fixed probability are called mixed strategies. Thus, there is a probabilistic situation and objective of the players is to maximize expected gains or to minimize expected losses by making choice among pure strategies with fixed probabilities.

SOLUTION OF GAMES

The games can be solved as follows:

1. Pure strategies
2. Mixed strategies

Games with saddle point [pure strategies]

- “Saddle point of a payoff matrix is an equilibrium point or common point of row and column”
- “If the maximin value equals to minimax value, then the game is said to have a saddle point”.

Rules to determine saddle point

- a. Select the minimum [lowest] element in each row of the payoff matrix and write them under ‘row minima’ heading. Then select the largest element among these elements and enclose it in a rectangle, ‘ $\square$ ’.
- b. Select the maximum [largest] element in each column of the payoff matrix and write them under ‘column maxima’ heading. Then select the lowest element it in a circle, ‘ $\bigcirc$ ’.
- c. Find out $\bigcirc$ the element(s) which is same in the rectangle as well as circle and mark the position of such elements(s) in the matrix. This element represents the value of the game and is called the saddle (or) equilibrium point.

Games saddle point [mixed strategies]

In certain cases, there is no pure strategy solution for a game, i.e., no saddle point exists. In all such cases, to solve games both the players must determine an optimal mixture of strategies to find a saddle point.

The optimal strategy mixture for each player may be determined by assigning to each strategy its probability of being selected. The strategies so determined are called mixed strategies because they are probabilistic combination of available choices by strategy.

The value of game obtained by the use of mixed strategies represents best payoff which player A can expect to win and the least which player B can lose. The expected payoff to a player in a game with arbitrary payoff matrix $[a_{ij}]$ of order $m \times n$ is defined as:

$$E(p,q) = \sum_{i=1}^m \sum_{j=1}^n p_i a_{ij} q_j$$

$$= P^T A Q \text{ [in matrix notation]}$$

where,

P = Player A's strategies, $S(A) = \{ p_1, p_2, \dots, p_m \}$

Q = Player B's strategies, $S(B) = \{ q_1, q_2, \dots, q_m \}$

Note of Book:

- $p_1 + p_2 + \dots + p_m = 1$
- $q_1 + q_2 + \dots + q_m = 1$

- A player chooses a particular strategy with particular probability can also which be interpreted as the relative frequency with a strategy is chosen from the number of strategies available to a player in a particular game.

Rules to determine saddle point

If the saddle point do not exists in a given $m \times n$ payoff matrix, then the following rules are used to reduce the matrix into 2×2 .

- In a $m \times n$ payoff matrix, if each element in a row, say R_d , is less than or equal to the corresponding element in another row, say R_s , then the row R_d is said to be dominated by row R_s and therefore, row R_d can be deleted from the payoff matrix.
- In a $m \times n$ payoff matrix, if each element in a column, say C_d , is greater than or equal to the corresponding element in another column, say C_s , then the column C_d is said to be dominated by column C_s and therefore, column C_d can be deleted from the payoff matrix.
- In a $m \times n$ payoff matrix, if the domination is strict, then the rows and/or columns are deleted by average method as per the rules 'a' and 'b'.

2X2 matrix:

The $m \times n$ matrix can be reduced as 2×2 matrix and the payoff values are written as follows:

2X2 matrix

		Player B	
		B1	B2
Player A	A1	a_{11}	a_{12}
	A2	a_{21}	a_{22}

Or

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Note of Book:

- In 2×2 matrix the optimal strategies for player A and B are given as follows:
 Player A's strategies, $S(A) = \{p_1, p_2\}$
 Player B's strategies, $S(B) = \{q_1, q_2\}$

Methods of mixed strategies

A mixed strategy game can be solved by different solution methods such as:

- Arithmetic or short-cut method
- Graphical method
- Dominance property

Arithmetic method

This method is also called *short-cut method*. It provides an easy procedure for finding optimal strategies for each player in a 2X2 matrix without saddle point.

The following steps are used under this method:

- Find the differences between the two values in the first row and put it against second row of the matrix, neglecting negative sign (if any).
- Find the differences between the two values in the second row and put it against first row of the matrix, neglecting negative (if any).
- Repeat step 'a' and 'b' for the two columns also.

Note of Book:

- The value obtained by 'swapping the differences' represents the optimal relative frequencies of play for both players' strategies.
- d. The frequencies are converted into probabilities by dividing each of them by their sum.
- e. The value of the game can be obtained as follows:

$$V = \frac{a_{11}a_{22} - a_{12}a_{21}}{[a_{11} + a_{22}] - [a_{12} + a_{21}]}$$

Note of Book:

- The value of the game can also be obtained from any one of the following equations:
 - $V = a_{11}p_1 + a_{21}p_2$
 - $V = a_{12}p_1 + a_{22}p_2$
 - $V = a_{11}q_1 + a_{12}q_2$
 - $V = a_{21}q_1 + a_{22}q_2$

Note of Book:

- The player B strategies are represented in column vector.

Graphical method

The graphical method is useful for the game where the payoff matrix is of the size '2Xn' or 'mX2' i.e., the game with mixed strategies that has only two un dominated pure strategies for one of the players in the two-person zero-sum game.

Optimal strategies for both the player assign non-zero probabilities to the same number of pure strategies. Therefore, if one player has only two strategies, the other will also use the same number of strategies. Hence, this method is useful in finding out which of the two strategies can be used.

Case:1: 2Xn matrix is given:

Consider the following 2Xn payoff matrix of a game without saddle point.

		Player B				Probability
		B_1	B_2		B_n	
Player A	A_1	a_{11}	a_{12}		a_{1n}	p_1
	A_2	a_{21}	a_{22}		a_{2n}	p_2

Probability	q_1	q_2	$\dots q_n$

Player A has two strategies A_1 and A_2 with probability of their selection p_1 and p_2 , respectively, such that $p_1 + p_2 = 1$ and $p_1, p_2 \geq 0$. Now for each of the pure strategies available to player-B, expected payoff for player A would be as follows:

B's pure strategies	A's expected payoff
B_1	$a_{11}p_1 + a_{21}p_2$
B_2	$a_{12}p_1 + a_{22}p_2$
.	.
.	.
.	.
B_n	$a_{1n}p_1 + a_{2n}p_2$

According to the maximin criterion for mixed strategy games, player A should select the value of probability p_1 and p_2 so as to maximize his minimum expected payoffs. This may be done by plotting the straight lines representing player A's expected payoff values.

The highest point on the lower boundary of these lines will give maximum expected payoff among the minimum expected payoffs and the optimum value of probability p_1 and p_2 .

Now the two strategies of player B corresponding to those lines which pass through the maximin point can be determined. It helps in reducing the size of the game to (2X2), which can be easily solved by any of the methods discussed earlier.

Case:2: $m \times 2$ matrix is given:

Consider the following $m \times n$ payoff matrix of a game without saddle point.

		Player B		Probability
		B_1	B_2	
Player A	A_1	a_{11}	a_{12}	p_1
	A_2	a_{21}	a_{22}	p_2
	.	.	.	.
	.	.	.	.
	.	.	.	.
	A_m	a_{m1}	a_{m2}	p_m
Probability		q_1	q_2	

Player B has two strategies B_1 and B_2 with probability of their selection q_1 and q_2 , respectively, such that $q_1 + q_2 = 1$ and $q_1, q_2 \geq 0$. Now for each of the pure strategies available to player-A, expected payoff for player B would be as follows:

A's pure strategies	B's expected payoff
---------------------	---------------------

A_1	$a_{11}q_1 + a_{12}q_2$
A_2	$a_{21}q_1 + a_{22}q_2$
$\cdot$	$\cdot$
$\cdot$	$\cdot$
$\cdot$	$\cdot$
A_m	$a_{m1}q_1 + a_{m2}q_2$

According to the minimax criterion for mixed strategy games, player B should select the value of probability q_1 and q_2 so as to minimize his maximum expected payoffs. This may be done by plotting the straight lines representing player B's expected payoff values.

The lowest point on the upper boundary of these lines will give minimum expected payoff among the maximum expected payoffs and the optimum value of probability q_1 and q_2 .

Now the two strategies of player A corresponding to those lines which pass through the minimax point can be determined. It helps in reducing the size of the game to (2X2), which can be easily solved by any of the methods discussed earlier.

Dominance property:

In some of the games, it is possible to reduce the size of the payoff matrix by eliminating rows (or columns) which are dominated by other rows (or columns) respectively.

Dominance property for rows: $X \leq Y$ i.e. if all the elements of a particular row **X** are less than or equal to the corresponding elements of another row **Y** then delete row **X** (row **X** is dominated by row **Y**). The elements of a particular row **X** can also be compared with an average of two or more other rows and if the elements of the row **X** are less than or equal to the corresponding elements after taking the average then delete row **X**.

Dominance property for columns: $X \geq Y$ i.e. if all the elements of a particular column **X** are greater than or equal to the corresponding elements of another column **Y** then delete column **X** (column **X** is dominated by column **Y**). The elements of the column **X** can also be compared with an average of two or more columns and if the elements of column **X** are greater than the corresponding elements after taking the average then delete the column **X**.

UNIT-V

QUEUING THEORY

INTRODUCTION

A common situation occurring or happening in every life is that of queuing or waiting in a line.

Queuing [waiting line] are usually seen at bus stops, ticket booths, doctor's clinics, bank counters, and traffic lights and so on.

Queues are also found in workshops where the machines wait to be repaired, in warehouse where items are wait to b used, incoming calls wait to mature in the telephone exchange, trucks wait to be unloaded, airplanes wait either to takeoff or land and so on.

These problems arising due to:

- a. Too much demand on the facilities.
- b. Number of customers exceeds to the number of service facilities.
- c. Service facilities do not work efficiently.
- d. Server takes more time than prescribed time to serve a customer.

Queuing theory can be applied to a variety of operational situations where it is not possible to predict accurately the arrival rate (or time) of customers and service rate (or time) of service facility or facilities.

In practical, it can be used to determine the level of service [either the service rate or the number of service facilities] that balances the following two conflicting costs:

- a. Cost of offering the service.
- b. Cost incurred due to delay in offering service.

The first cost is associated with the service facilities and their operation, and the second represents the cost of customer's waiting time.

Many practical situations in which study of queuing theory can provide solution to waiting line problems are listed in the following table:

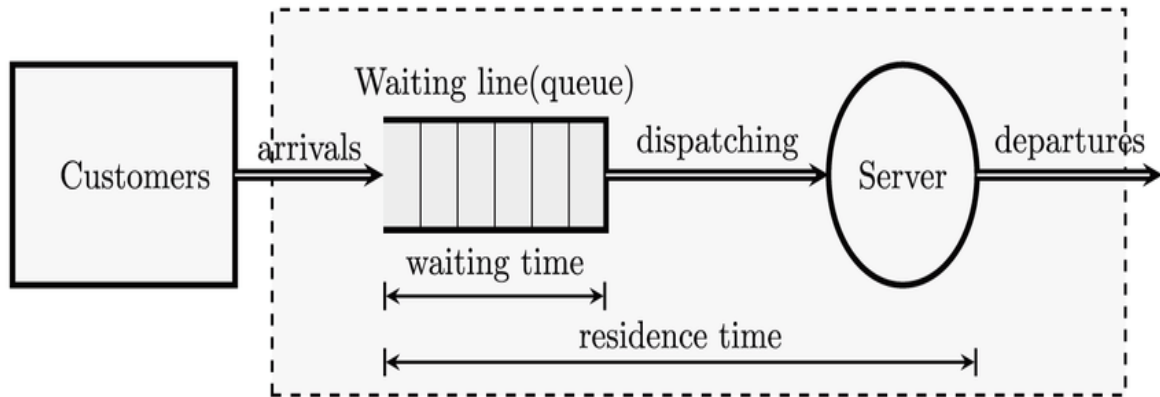
Examples of queues

Station	Customers	Service facility
1. Petrol station	Automobiles	Petrol pumps
2. Hospital	Patients	Doctors/Nurses/Rooms
3. Post office	Letters	Sorting system
4. Airport	Aircraft	Runways
5. Job interview	Applicants	Interviewers

QUEUING STRUCTURE/SYSTEM

A queuing system is composed of the following components or parts:

1. Calling population/input source
2. Queuing process/arrival process
3. Queue discipline
4. Service process



COMPONENTS OF QUEUING STRUCTURE

Calling population or Input source

Customers are defined as those in need of service. Customers requiring service are generated at different times by “*calling population*”, commonly known as “*input source*”.

The arrivals or inputs to the system are characterised by:

- a. Size of calling population
- b. Behaviour of the arrivals
- c. Pattern of arrivals at the system

Size of calling population

The size of calling population whether it is homogeneous or consists of several subpopulations is considered to be either finite (limited) or infinite (unlimited).

If the customers arrival depends on the number of customers already in the system (in service plus in queue), the calling population is called limited or finite.

Example:

A factory having only four machines which often require repair/service and two of them (say) are in working condition. Then at any point in time, there are only two machines which could possibly require service.

Alternatively, if new customer’s arrival is independent of the number of customers already in the system, the calling population is called unlimited or infinite.

Example:

The customers arriving at a bank or super market, students arriving to get admission at the university, cars arriving at a highway petrol station, etc.

Behaviour of arrivals

If a customer, on arriving at the service system wait in the queue until served and do not switch between waiting lines is called a *patient customer*.

Example:

Machines arrived at the maintenance shop

Whereas the customer, who waits for a certain time in the queue and leaves the service system without getting service due to certain reasons is called an impatient customer.

Example:

A customer who has just arrived at a grocery store and finds that the salesmen are busy in serving the customers already in the system, will either wait for service till his patience is exhausted or estimates that his waiting time may be excessive and so leaves immediately to seek service elsewhere.

Key situations in arrivals

Balking: Customers do not join the queue either by seeing the number of customers already in service system or estimating the excessive waiting time for desired service.

Reneging: Customers after joining the queue, wait for sometimes in the queue but leave before being served on account of certain reasons.

Jockeying: Customers move from one queue to another hoping to receive service more quickly (a common scene at railway booking window).

Pattern of arrivals at the system

Customers may arrive in batches (such as the arrival of a family at a restaurant) or individually (such as the arrival of a train at a platform). These customers may arrive at a service facility either on scheduled time (by prior information) or unscheduled time (without information).

The arrival process (or pattern) of customers to the service system is classified into two categories: *static* and *dynamic*.

These two are further classified based on the nature of arrival time/rate and the control which can be exercised on the arrivals.

In *static arrival pattern*, the control depends on the nature of arrival rate (random or constant). In random (or unscheduled) arrivals the times are random variable and therefore we need to understand the average and frequency distribution of the times.

The *dynamic arrival pattern*, controlled by both service facility and customers. The service facility adjusts its capacity to match changes in the service intensity, by either varying the staffing levels at different timings of service, varying service charges (such as telephone call charges at different hours of the day or week) at different timings, or allowing entry with appointments.

The arrival time distribution can be approximated by one of the following probability distributions:

- a. Poisson distribution
- b. Exponential distribution

The *Poisson distribution* is a discrete probability that provides probabilities for the number of customers that may arrive in any specific interval time, whereas *exponential distribution* provides probabilities for times gap between two consecutive arrivals.

Probability functions:

Let n customers arrive during a time interval '0' to 't'. If ' λ ' is the expected (or average) number of arrivals per unit time, then

- a. The Poisson expected number of arrivals will be λ
- b. The exponential expected number of arrivals will be $\frac{1}{\lambda}$

Queuing process or Arrival process

The way in which customers arrive at the service facility individually or in batches at scheduled or unscheduled time is called the "*arrival process*".

The queuing process refers to the number of queues, and their respective lengths. The number of queues may be single, multiple or priority queues depend upon the layout of a service system. The length (or size) of the queue depends upon the operational such as physical space, legal restrictions and attitude of the customers.

In certain cases, a service system is unable to accommodate more than the required number of customers at a time. No further customers are allowed to enter until space becomes available to accommodate new customers. Such types of situations are referred to as finite (or limited) source queue.

Example:

- Cinema halls, restaurants, etc.

On the other hand, if a service system is able to accommodate any number of customers at a time, then it is referred to as infinite (or unlimited) source queue.

Example:

- In a sales department where the customer orders are received, there is no restriction on the number of orders that can come in so that a queue of any size can form.

In many other situations, when arriving customers found long queue(s) in front of a service facility, they often do not enter the service system even though additional waiting space is available. The queue length in such cases depends upon the attitude of the customers.

Example:

- When a motorist finds that there are many vehicles waiting at the petrol station, in most of the cases he does not stop at this station and seeks service elsewhere.

Queue discipline

The way of selecting customers for service according to certain rules known as the "*queue discipline*".

There are number of ways in which customers in the queue are served. Some of these are:

- a. Static queue discipline
- b. Dynamic queue discipline

Static queue discipline

These are based on the individual customer's status in the queue. Few of such disciplines are:

First come first served (FCFS): If the customers are served in the order of their arrival, then this is known as the first-come first served queue discipline.

Example:

- Hospitals, petrol stations.

Last come first served (LCFS): In this discipline the last customer arrived is served first because it reduces the time.

Example:

- Cargo, movers and packers, production process.

Dynamic queue discipline

These are based on the individual customer attributes in the queue. Few of such disciplines are:

Service in Random Order (SIRO): Under this rule customers are selected for service at random irrespective of their arrivals in the service system.

Priority Service (PS): Under this rule customers are grouped in priority classes on the basis of some attributes such as service time or urgency and FCFS rule is used within each class to provide service.

Example:

- The payment of telephone or electricity bills by cheque or cash.

Pre-Emptive Priority or Emergency (PEP): Under this rule, an important customer is allowed to enter into the system even if a customer with lower priority is already in service.

Non- Pre-Emptive Priority (NPEP): In this case an important customer goes ahead in the queue, but service is started immediately on the completion of the current service.

Service Process

The way of providing service to the customers is called “*service process*”.

It is to be by one or more service facilities, this may be person (bank teller) or space (airport runway, parking).

Service facility may be without server (self service) or include one (single server) or more servers operating either in series as a team or in parallel (multiple servers).

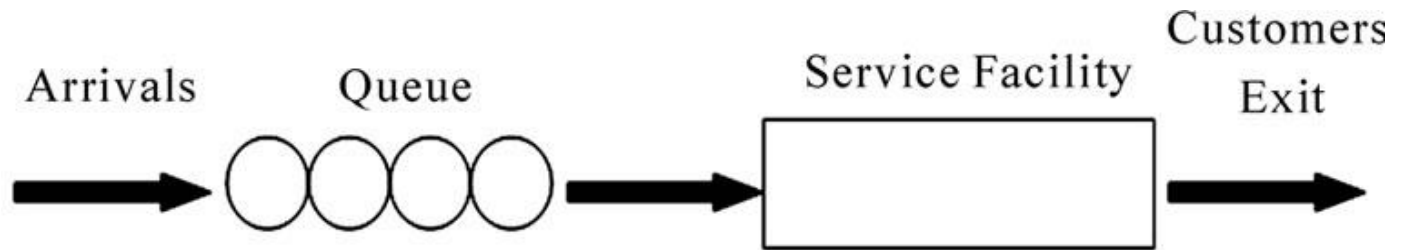
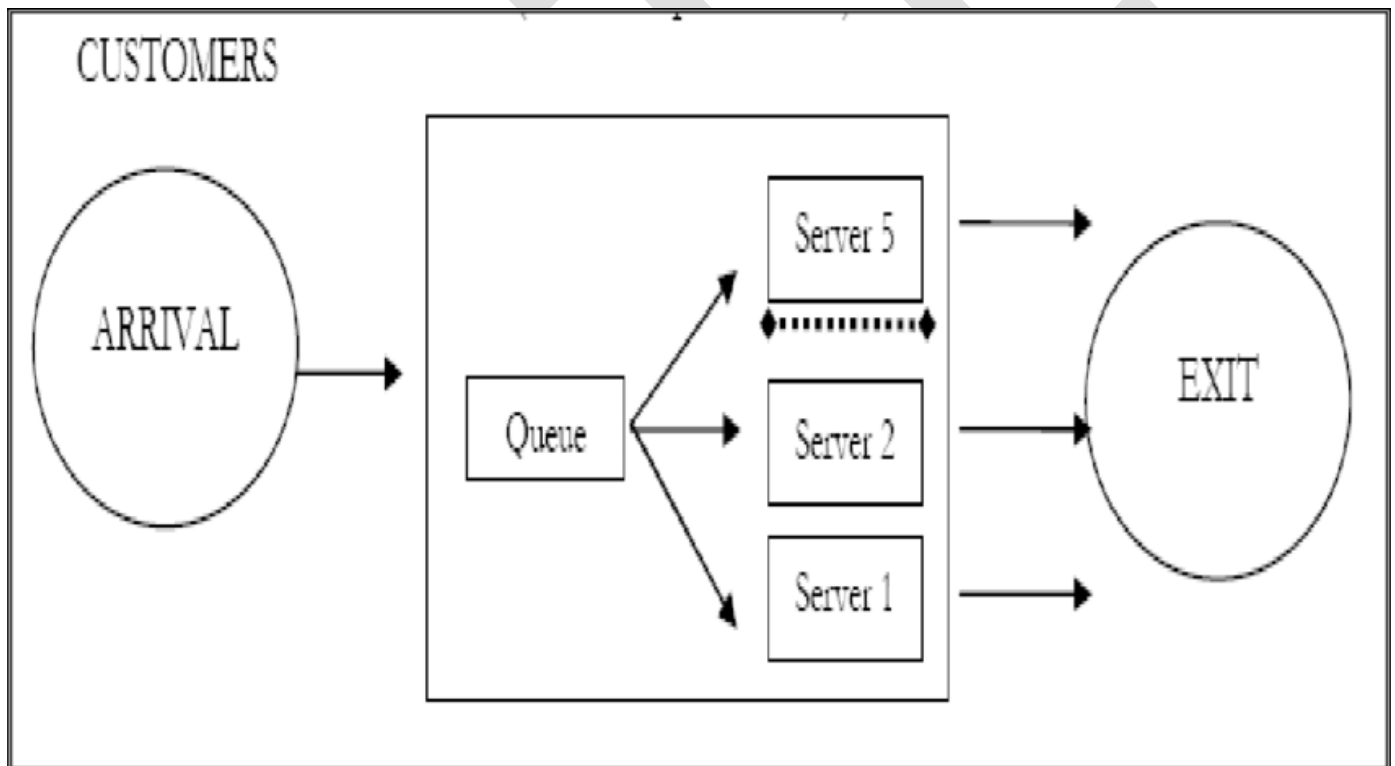
The service process (mechanism) is concerned with the manner in which customers are serviced and leave the service system. It is characterised by two ways:

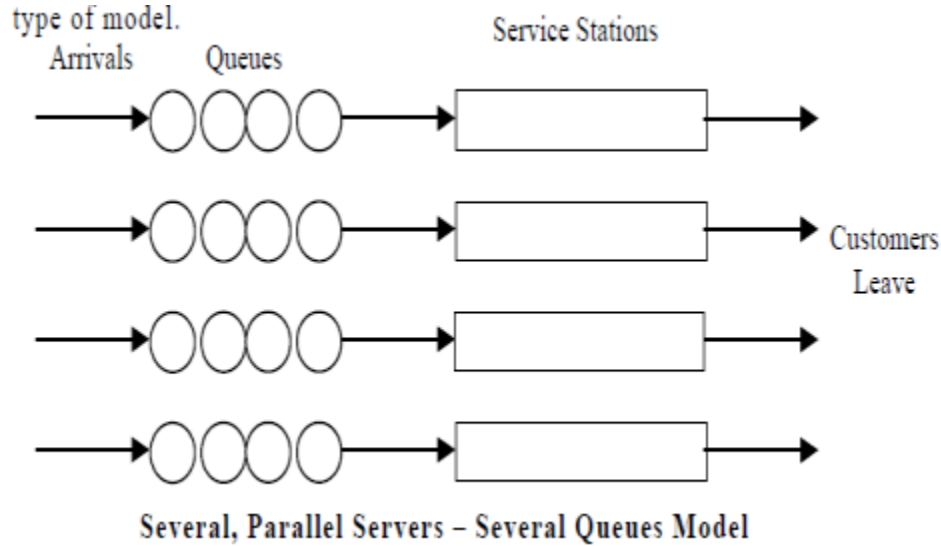
- a. Arrangement of service facilities

b. Service time distribution

Arrangement of service facilities

The capacity (arrangement) of the service facility is measured in terms of customers that can be served simultaneously and/or effectively. The service facilities (or servers), commonly known as service channels may be in series or in parallel or mixed (partly in series and partly in parallel) as follows:

Single queue, single service facility:***Single queue, multiple service facilities (parallel):***

Multiple queues, multiple service facilities (parallel):**Service time distribution**

The time taken by the server from the commencement of service to the completion of service for a customer is known as the service time.

The service time can be approximated by one of the following probability distributions:

- i. Poisson distribution
- ii. Exponential distribution

Let n customers arrive during a time interval '0' to 't'. If ' μ ' is the expected (or average) service rate per unit time, then

- a. The Poisson expected number of arrivals will be μ
- b. The exponential expected number of arrivals will be $\frac{1}{\mu}$

Notations in Queuing Theory

The notations used in the analysis of a queuing system are as follows:

1. n = number of customers in the system (waiting and in service)
2. N = maximum number of customers allowed in the system
3. s = number of service channels (service facilities of servers)
4. q = waiting line or queue
5. λ = average (or expected) customer arrival rate per unit of time.
6. μ = average (or expected) customer service rate per unit of time.
7. ρ = traffic intensity (or) service utilization factor (or) average (or expected) fraction of time for which server is busy.

$$= \frac{\text{average service completion time} \left[\frac{1}{\mu} \right]}{\text{average interarrival time} \left[\frac{1}{\lambda} \right]} = \frac{\lambda}{\mu}$$

8. P_0 = probability of no customer in the system = $(1 - \rho)$
9. P_n = probability of 'n' customer in the system
10. L_s = average (or expected) number of customers in the system
11. L_q = average (or expected) number of customers in the queue
12. W_s = average (or expected) waiting time in the system
13. W_q = average (or expected) waiting time in the queue

SOLUTION OF QUEUING MODELS

Two approaches are available for solving queuing models viz., mathematical and simulation.

In mathematical approach the actual arrival and service time distributions are approximated by one of the known mathematical distributions, i.e., Poisson, exponential, etc., to describe expected value of various operational characteristics of the queuing process.

In this section, we will discuss various queuing models in which both arrivals and departures (those served) take place simultaneously with specific assumptions about the features of queuing systems such as: input source, arrival process, queue configuration, queue discipline and service process.

The simultaneous occurrence of arrivals and departures is also called *birth and death process*. This process helps in determining the probability distribution of the number of customers in the queuing system at a particular instance of time. The probability distribution so obtained is then used to determine performance measures (operating characteristics) of various types of queuing models.

Types of models

1. Single-Server Queuing Models
 - a. Model-I: $\{(M/M/1): (\infty/FCFS)\}$ - Unlimited Queue
 - b. Model-II: $\{(M/M/1): (N/FCFS)\}$ - Limited Queue
2. Multiple-Server Queuing Models
 - a. Model-III: $\{(M/M/s): (\infty/FCFS)\}$ - Unlimited Queue
 - b. Model-IV: $\{(M/M/s): (N/FCFS)\}$ - Limited Queue

Single-Server Queuing Models

Model-I: $\{(M/M/1): (\infty/FCFS)\}$ - Unlimited Queue

This model is based on certain assumptions about the queuing system:

1. Arrivals are described by Poisson probability distribution and come from an infinite population.
2. Single waiting line and each arrival waits to be served regardless of the length of the queue and no balking or reneging.
3. Queue discipline is "FCFS".
4. Single server and service time follow exponential distribution.
5. The average service rate is more than the average arrival rate.

If the probability of the system being empty (no customer) is denoted by P_0 and written as: $P_0 = 1 - \rho$

then, the probability of n customers in the system is denoted by P_n and written as:

$$P_n = \rho^n P_0$$

Performance measures:

1. $L_s = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$
2. $L_q = \rho \times L_s = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$
3. $W_s = \frac{L_s}{\lambda} = \frac{1}{\mu-\lambda}$
4. $W_q = \frac{L_q}{\lambda} = W_s \times \rho = \frac{\lambda}{\mu(\mu-\lambda)}$

5. Probability that number of customer, 'n' in a system exceeds on given number 'k' as written as:

$$P(n \geq k) = \rho^k$$

Model-II: {(M/M/1): (N/FCFS)}-Limited Queue

In this model no more than N customers can be accommodated at any time in the system due to certain reasons. Thus any customer arriving when the system is already contains N customers does not enter the system and is lost.

Example: A finite queue may be arising due to physical constraint such as:

1. Emergency room in a hospital
2. One man barber shop with certain number of chairs for waiting customers etc.

Assumptions of this model are same as that of Model-I except that the length of the queue be limited.

If the number of customers $n=0,1,2,\dots,N$ (or) $n \leq N$

then,

the probability of the system being empties (no customer) is denoted by P_0 and written as:

$$P_0 = \frac{1-\rho}{1-\rho^{(N+1)}}$$

the probability of n customers in the system is denoted by P_n and written as:

$$P_n = \rho^n P_0 ; \rho \neq 1$$

Performance measures:

1. $L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{(N+1)}}{1-\rho^{(N+1)}}$
2. $L_q = L_s - \rho$
3. $W_s = \frac{L_s}{\lambda[1-P_N]}$
4. $W_q = W_s - \frac{1}{\mu} = \frac{L_q}{\lambda[1-P_N]}$

Where; $P_N = \rho^N P_0$

Multiple-Server Queuing Models

Model-III: $\{(M/M/s): (\infty/FCFS)\}$ -Unlimited Queue

In this case instead of single server, there are multiple but identical servers in parallel to handle arriving customers.

For this queuing system, it is assumed that arrivals follow a Poisson probability distribution at an average rate of λ customers per unit of time and are served on a first-come, first served basis by any of the servers. The service times are distributed exponentially with an average of μ customers per unit of time. It is further assumed that only one queue is formed.

If the probability of the system being empty (no customer) is denoted by P_0 and written as:

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n + \frac{1}{s!} \left(\frac{\lambda}{\mu} \right)^s \left(\frac{s\mu}{s\mu - \lambda} \right) \right]^{-1}$$

Then; the probability of n customers in the system is denoted by P_n and written as:

$$P_n = \frac{\rho^n P_0}{n!}; \rho = \frac{\lambda}{s\mu}$$

Performance measures:

1. $L_q = \left[\frac{(\lambda\mu) \left(\frac{\lambda}{\mu} \right)^s}{(s-1)! (s\mu - \lambda)^2} \right] P_0 = L_s - \frac{\lambda}{\mu}$
2. $L_s = L_q + \frac{\lambda}{\mu}$
3. $W_q = \frac{L_q}{\lambda}$
4. $W_s = W_q + \frac{1}{\mu}$
5. Probability that all servers are simultaneously busy (utilization factor)

$$P(n \geq k) = \left[\frac{1}{s!} \left(\frac{\lambda}{\mu} \right)^s \left(\frac{s\mu}{s\mu - \lambda} \right) \right] P_0$$

Model-IV: $\{(M/M/s): (N/FCFS)\}$ -Limited Queue

If the probability of the system being empty (no customer) is denoted by P_0 and written as:

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n + \sum_{n=s}^N \left(\frac{1}{s!s^{(n-s)}} \right) \left(\frac{\lambda}{\mu} \right)^n \right]^{-1}$$

Then; the probability of n customers in the system is denoted by P_n and written as:

$$P_n = \left[\sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n \right] P_0; \rho = \frac{\lambda}{s\mu}$$

Performance measures:

1. $L_q = \left[\frac{(s\rho)^s \rho}{s!(1-\rho)^2} \right] \left[1 - \rho^{(N-s+1)} - (1-\rho)(N-s+1)\rho^{(N-s)} \right] P_0$
2. $L_s = L_q + \frac{\lambda}{\mu} [1 - P_N]$

$$3. W_q = \frac{L_q}{\lambda[1-P_N]}$$

$$4. W_s = \frac{L_s}{\lambda[1-P_N]}$$

where; $P_N = P_n$